International Journal of Computer Information Systems and Industrial Management Applications
ISSN 2150-7988 Volume 18(2s) 2026 pp. 1291-1301
https://doi.org/10.70917/ijcisim-2026-2095
Article

HYBRID SWEEP TECHNIQUES IN CAPUTO IMPLICIT SCHEMES: A NOVEL FRAMEWORK FOR SPACE-FRACTIONAL PDES STABILITY AND EFFICIENCY

Rina Julita[1]*, Midhun Chakkaravarthy[2], Andang Sunarto[3]

[1]School of AI Computing and Multimedia, Lincoln University College, Malaysia
[2]School of AI Computing and Multimedia, Lincoln University College, Malaysia
[3]Fatmawati Islamic University, Indonesia
Corresponding Author: Rina Julita (Email: rinajulita@unived.ac.id)

Abstract: This study presents an advanced numerical framework for the efficient and stable solution of Space-Fractional Partial Differential Equations (SFPDEs), which are widely used to model anomalous transport and diffusion processes in complex systems. The proposed method integrates Caputo's implicit discretization with novel hybrid sweep strategies—namely full-sweep (FS), half-sweep (HS), quarter-sweep (QS), and a newly introduced adaptive quarter-sweep (AQS) scheme. Each sweeping approach is designed to reduce computational burden while maintaining numerical accuracy and stability.To further enhance convergence performance, the framework incorporates preconditioned iterative solvers, specifically the Preconditioned Successive Overrelaxation (PSOR) and Preconditioned Accelerated Overrelaxation (PAOR) methods. These solvers modify the spectral properties of the underlying system matrix, thereby accelerating convergence without compromising solution quality.Comprehensive numerical experiments are conducted on one- and two-dimensional benchmark SFPDEs, spanning matrix sizes from 128 to 2048. Results consistently demonstrate the superiority of the AQS-PAOR combination in terms of convergence speed, computational efficiency, and stability under long-time integration. For large-scale systems, the proposed scheme achieves up to 35% reduction in execution time compared to traditional QS-PSOR methods, with only marginal differences in error norms.The framework offers scalability, robustness, and potential applicability to higher-dimensional or nonlinear SFPDEs, making it suitable for real-time simulation and embedded computing environments. This work bridges the gap between classical integer-order sweeping techniques and fractional-order numerical schemes, setting the stage for further advances in adaptive solvers for complex dynamical systems.....

Keywords: Space-fractional PDEs, Caputo derivative, hybrid sweep, PSOR, PAOR, adaptive algorithm, numerical stability

1. INTRODUCTION

Space-Fractional Partial Differential Equations (SFPDEs) have emerged as powerful mathematical tools for modeling complex diffusion and transport phenomena that deviate from classical Brownian motion Hoe and Hasan (2013). Such anomalous behavior is commonly encountered in heterogeneous porous media, turbulent flows, viscoelastic materials, and financial systems with memory effects (Valerio et al., 2013). The use of fractional derivatives, especially in the spatial domain, allows for a more realistic representation of long-range interactions and memory-dependent dynamics, which traditional integer-order models often fail to capture (Podlubny, 1999; Kilbas et al., 2006).

Among the different definitions of fractional derivatives, the Caputo derivative has gained popularity for its compatibility with physically meaningful initial conditions, particularly in initial-boundary value problems Atangana and Baleanu (2013). Unlike the Riemann–Liouville derivative, the Caputo form incorporates integer-order initial values, thus simplifying physical interpretation and numerical implementation (Podlubny, 1999). However, the discretization of SFPDEs using Caputo's definition leads to non-local and dense linear systems, especially when implemented over fine spatial grids or long temporal intervals. These systems pose challenges in terms of computational cost, memory usage, and stability.

Classical iterative solvers such as Full Sweep Point Iteration (FSPI) or Gauss–Seidel often suffer from slow convergence when applied to such systems, especially as the size of the matrix increases. In response to these limitations, various sweeping techniques—such as half-sweep (HS) and quarter-sweep (QS) strategies—have been introduced in integer-order PDEs to reduce computational overhead by selectively updating subsets of the solution grid (Sulaiman et al., 2004a, 2004b). While effective in some applications, these techniques have rarely been extended to the fractional domain, where the inherent non-locality complicates update dependencies and convergence behaviors (Yang et al., 2008).

Furthermore, the convergence rate of iterative solvers can be significantly improved using preconditioning techniques, which transform the system into a more favorable form for iteration (Young, 1971; Conte and de kasar, 1980; Watkins, 2002).. Methods such as the Preconditioned Successive Overrelaxation (PSOR) and Preconditioned Accelerated Overrelaxation (PAOR) have been proposed to accelerate convergence by altering the spectral radius of the system matrix (Young, 1971). When combined with sweep strategies, these preconditioners can provide significant gains in both accuracy and computational speed, making them promising candidates for fractional PDE solvers.

This study addresses the gap between hybrid sweep techniques and fractional-order discretizations by proposing a unified computational framework that integrates Caputo implicit schemes with full-, half-, quarter-, and adaptive quarter-sweep (AQS) strategies, evaluated under PSOR and PAOR solvers. The approach is validated through extensive numerical experiments on 1D and 2D SFPDEs, examining iteration count, error norms, and execution time across various matrix sizes. The findings not only demonstrate the superior efficiency of the AQS-PAOR combination but also open the door for applying these techniques in high-dimensional and real-time simulation contexts.

2. LITERATURE REVIEW

The numerical treatment of space-fractional partial differential equations (SFPDEs) has evolved significantly in recent decades, owing to their widespread application in modeling anomalous diffusion, wave propagation in complex media, and nonlocal dynamics Yang *et al.*'s research (2008). Various schemes have been proposed for both time-fractional and space-fractional operators, with a key distinction between local and nonlocal approximation methods.

2.1 Fractional Derivatives and Discretization Techniques

Podlubny (1999) provided a foundational framework for fractional calculus, introducing definitions such as the Riemann–Liouville and Caputo derivatives. The Caputo derivative, in particular, is favorable in physical problems due to its use of standard initial conditions, making it suitable for initial-boundary value problems. Several studies (Kilbas et al., 2006; Meerschaert & Tadjeran, 2004) have adopted finite difference methods, including shifted Grünwald–Letnikov approximations, for discretizing fractional derivatives, though these approaches often result in full matrices and high memory demands.

Spectral methods have also been explored to enhance accuracy (Zayernouri & Karniadakis, 2013), though they are often limited to smooth solutions and require global basis functions. Discontinuous Galerkin and finite element approaches (Xu & Hesthaven, 2014) offer flexibility in complex geometries but introduce greater algorithmic and implementation complexity.

2.2 Iterative Solvers for Fractional Systems

Due to the dense and ill-conditioned nature of the resulting matrices, iterative solvers are typically preferred over direct methods. Young (1971) first formalized Successive Overrelaxation (SOR) as an improvement over Gauss–Seidel for solving large linear systems. Later studies by Abdullah (1991) and Sulaiman et al. (2004) extended these to include **half-sweep** and **quarter-sweep** improvements that significantly reduced computational overhead in elliptic PDEs.Preconditioning techniques have also been a focus of study. The Preconditioned SOR (PSOR) and its accelerated variant PAOR have been applied in both classical PDEs and fractional cases (Yang et al., 2008), yielding faster convergence when used with suitable sweep strategies. However, few works combine preconditioned solvers with adaptive sweeping in the fractional domain.

2.3 Hybrid and Adaptive Sweeping Methods

Hybrid sweeping methods have been successfully applied in solving elliptic and parabolic PDEs (Sulaiman et al., 2008), but their application to SFPDEs is relatively new. The challenge lies in the non-local structure of fractional operators, which can render conventional sweep techniques inefficient or unstable if not carefully adapted.Recent studies (Zhuang & Liu, 2006; Valerio et al., 2013) have pointed to the necessity of tailoring solvers to the unique spectral structure of fractional matrices. However, a systematic evaluation of quarter- and adaptive-sweep strategies within the Caputo implicit framework—particularly when coupled with PAOR or PSOR—remains largely unexplored.

This study addresses this gap by integrating hybrid and adaptive sweeping with preconditioned iterative solvers, and empirically evaluating their efficiency, stability, and scalability across multiple SFPDE benchmarks.

3. METHODOLOGY

This study presents a finite difference method framework that integrates Caputo's fractional derivative with a set of novel sweeping strategies—namely, Full Sweep (FS), Half Sweep (HS), and Quarter Sweep (QS)—for efficiently solving one-dimensional space-fractional parabolic partial differential equations (SFPDEs). The methodology is designed to enhance numerical stability, accuracy, and computational performance. Furthermore, iterative solvers such as the Preconditioned Successive Over-Relaxation (PSOR) and Preconditioned Accelerated Over-Relaxation (PAOR) methods are incorporated to optimize the solution of the resulting sparse linear systems.

Mathematical Model

Consider the following one-dimensional space-fractional parabolic PDE involving the **Caputo fractional derivative**:

$$\frac{\partial u}{\partial t} = a(x)\frac{\partial^{\beta} u}{\partial x^{\beta}}, \quad x \in [\rho_0, \rho_1], 0 \le t \le T$$

with initial conditions

$$u(x,0) = g_1(x), \quad x \in [\rho_0, \rho_1]$$

and boundary conditions

$$u(\rho_o, t) = g_2(t)$$

$$u(\rho_1, t) = g_3(t).$$

Furthermore, $a(x), g_1(x), g_2(t)$ dan $g_3(t)$ are known functions or constants.

Discretize the space-fractional derivative

We following description discusses the Grunwald-Letnikov fractional derivative operator approach (Samko *et al.,* 1987) that considers continuous functions $y = f(t)$. First, consider the definition of the first-order derivative shown as :

$$\frac{d}{dt}f(t) = \underset{h \to 0}{had}\frac{f(t) - f(t-h)}{h} \tag{3.1}$$

Furthermore, from equation (3.1), the definition of the *nth* level derivative can be obtained which is expressed as

$$\frac{d^n}{dt^n} f(t) = \underset{h\to 0}{had} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} f(t - kh), \tag{3.2}$$

with binomial coefficients defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} \tag{3.3}$$

By considering equation (3.2) and binomial coefficients (3.3) the value at each level *n* can be defined as

$$D^n f(x) = \underset{h\to 0}{had} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} f(x - kh), \tag{3.4}$$

The formula (3.4) can be made for non-integer values of *n* with $\alpha \in R$ he condition that the binomial coefficients must be expressed as Gamma functions to replace the standard factorial expression. The upper bound of the summation (non-integer, *n*) is defined as $\frac{t-\alpha}{h}$ (where *t* and α are the upper and lower bounds of differentiation, respectively). Furthermore, the general form of the Grunwald-Letnikov derivative operator can be defined as (Podlubny 1999; Loverro, 2004;)

$$D^\alpha f(x) = \underset{h\to 0}{had} \frac{1}{h^n} \sum_{k=0}^{\frac{t-\alpha}{h}} (-1)^k \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)} f(x - kh), \tag{3.5}$$

It can be explained that the definition of the Reimann-Liouville fractional integral can also be considered to define the fractional derivative. Therefore, equation (3.6) can be considered as an alternative fractional integral derivative operator.

With reference to equation (3.5), the discretization process of SFPDEs with the Grunwald-Letnikov derivative operator scheme has been done previously by Hilfler, (2000) and Jogdan *et al*., (2013). Therefore, with $\beta = 1$ and based on equation. The equation is obtained as (Hiffler, 2000; Jogdan *et al*., 2013):

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = \frac{1}{\Gamma(-\beta)} \underset{M\to\infty}{had} \frac{1}{h^\beta} \sum_{k=0}^{M} \frac{\beta(k-\beta)}{\Gamma(k+1)} u(x-(k-1)h, t) \tag{3.7}$$

where N is a positive integer, $h = \frac{(x_R - X_L)}{M}$ and $\Gamma(.)$ are Gamma functions.

This results in a system of linear equations for u^{n+1}, which needs to be solved iteratively.

Teknik Hybrid Sweep for System Optimization

This study implements three variations of sweep techniques on the Caputo scheme:

1. Full Sweep

Using all grid points in the spatial domain. The Caputo implicit scheme is formulated completely, resulting in a dense and accurate system of linear equations. The study of SFPDEs problems for the full sweep case with Caputo implicit discretization scheme has previously been carried out by Jogdan et al (2013).

2. Half Sweep

Utilizes half of the grid points. The objective is to reduce computational load without significant loss of accuracy. As explained by Abdullah (1991)

3. Quarter Sweep

Uses a quarter of the grid points. Used for extreme efficiency in large-scale problems. An application of the quarter sweep concept introduced by Othman and Abdullah (1999, 2000a, 2000b) has been used to define the Caputo derivative operator scheme of a quarter-sweep of a fractional-space expressed

Iterative Solvers

To solve the system of linear equations resulting from discretization, the prepared **Preconditioned One-Step Iterative Method** the entire system of linear equations can be described in general form as

$$A\underset{\sim}{u_n} = \underset{\sim}{f_n} \tag{3.8}$$

with

$$A = \begin{bmatrix} a_{\phi,\phi} & a_{2\phi,\phi} & a_{3\phi,\phi} & \cdots & a_{M-\phi,\phi} \\ a_{\phi,2\phi} & a_{2\phi,2\phi} & a_{3\phi,2\phi} & \cdots & a_{M-\phi,2\phi} \\ a_{\phi,3\phi} & a_{2\phi,3\phi} & a_{3\phi,3\phi} & \cdots & a_{M-\phi,3\phi} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{\phi,M-\phi} & a_{2\phi,M-\phi} & a_{3\phi,M-\phi} & \cdots & a_{M-\phi,M-\phi} \end{bmatrix}_{\left(\left(\frac{M}{\phi}\right)-1\right)x\left(\left(\frac{M}{\phi}\right)-1\right)}$$

$$\underset{\sim}{u_n} = \left[u_{\phi,n}\ u_{2\phi,n}\ u_{3\phi,n} \cdots u_{M-\phi,n}\right]^T$$

$$\underset{\sim}{f_n} = \left[f_{\phi,n}\ f_{2\phi,n}\ f_{3\phi,n} \cdots f_{M-\phi,n}\right]^T$$

while $\phi = 1,2,4$ the respective values refer to the full-sweep, half, and quarter cases.

Construction of PSOR Method Family Scheme

The family of preconditioned SOR (PSOR) iterative methods was discussed and initiated by Young, (1950); Gunawardena (1991), Li (2012) and Ndanusa and Adeboye (2012). This family of iterative methods is categorized as one of the families of one-step iterative methods. In fact, PSOR iteration methods can also be known as preconditioned Gauss-Seidel (PGS) methods with weighted parameters, ω which are actively used to accelerate the convergence rate for solving preconditioned linear systems of equations. To construct the formula family of the PSOR iteration method for the SFPDEs problem in the full, half, and quarter sweep cases, consider the coefficient matrix (3.53)which is redefined as

$$A^* = D - L - V \tag{3.55}$$

where *L, D* and *V* are the lower triangular, corner and upper triangular matrices, respectively. Therefore, the formulation of the family of PSOR methods with weighted parameters ω can be expressed as (Young, 1950; Gunawardena 1991, Li, 2012 and Ndanusa and Adeboye, 2012):

$$\underset{\sim}{x}^{(k+1)} = (D - \omega L)^{-1}\left[(1-\omega)D + \omega V\right]\underset{\sim}{x}^{(k)} + (D - \omega L)^{-1}\omega \underset{\sim}{f}^{*} \qquad (3.5\,6\,)$$

with ω and $\underset{\sim}{x}^{(k)}$ being the weighted parameter and unknown vector at iteration k, respectively. Algorithm 3.1 shows the implementation of the family of PSOR iterative methods namely FSPSOR, HSPSOR and QSPSOR on Caputo's preconditioned linear equation system (3.53), respectively. Figure 3.4 illustrates the implementation of the family of PSOR iterative methods.

Table 3.1: PSOR iteration method group description

No	Parameter Selection	Iterative method
1	$P = I,\ \omega \neq 1$	FSSOR, HSSOR, QSSOR
2	$P \neq I,\ \omega = 1$	FSPGS, HSPGS, QSPGS
3	$P = I,\ \omega = 1$	FSGS, HSGS, QSGS

Algorithm 3.1: Schematic of the PSOR iteration family

Assign prefix values ω, P, A^{*}, $\underset{\sim}{f}^{*}$, $\underset{\sim}{x_0}^{(0)} = 0$ and $\varepsilon \leftarrow 10^{-10}$.

For $n = 1,2,3,\ldots,N$ apply

For $i = p, 2p, 3p, \ldots, m-p,$ apply

$$\underset{\sim}{x}^{(k+1)} = (D - \omega L)^{-1}\left[(1-\omega)D + \omega V\right]\underset{\sim}{x}^{(k)} + (D - \omega L)^{-1}\omega \underset{\sim}{f}^{*}$$

Convergence Test $\left\|\underset{\sim}{x}^{(k+1)} - \underset{\sim}{x}^{(k)}\right\| \leq \varepsilon = 10^{-10}$. If converged, go to step (c). Otherwise repeat to step (a).

Calculate approximate value $\underset{\sim}{u_n} = P^{T}\underset{\sim}{x}$

Display numerical results.

Construction of PAOR Method Family Schema

In fact, preconditioned AOR (PAOR) one-step iterative methods have been performed by Wu *et al* (2007), Wang and Song (2009), Yun (2011) and Li (2012). Therefore, in this section, the formulation of a family of PAOR iterative methods namely FSPAOR, HSPAOR and QSPAOR for solving the system of conditional linear equations (3.5 3) in SFPDEs problems for the full, half and quarter sweep cases is discussed.

Look back the definition of the coefficient matrix A in equation (3.53). In fact, the formula construction of the PAOR point iteration method is the same as the PSOR iteration method family of generations (3.5). However, the PAOR iteration method involves the use of two different weighted parameters, namely ω and β. The formulation of the family of PAOR iteration methods can be expressed generally as (Hadjidimos, 1978; Hallet, 1986; Yeyios, 1989).

$$\underset{\sim}{x}^{(k+1)} = (D - \omega L)^{-1}\left[\beta V + (\beta - \omega)L + (1-\beta)D\right]\underset{\sim}{x}^{(k)} + \beta(D - \omega L)^{-1}\underset{\sim}{f}^{*} \qquad (3.5\,7\,)$$

The implementation of the PAOR iteration method family on the system of conditional linear equations (3.5 3) can be described in Algorithm 3.2 and illustrated in Figure 3.5. In addition, the implementation of other iteration schemes for the PAOR family of iteration methods can be summarized in Table 3.2.

Table 3.2: PAOR iteration method group description

No	Parameter Selection	Iterative method
1	$P = I, \omega \neq \beta \neq 1$	FSAOR,HSAOR, QSAOR
2	$P = I, \omega = \beta \neq 1$	FSSOR, HSSOR, QSSOR
3	$P = I, \omega = \beta = 1$	FSGS, HSGS, QSGS

Algorithm 3.2: PAOR iteration scheme.

Assign prefix values ω, β, P, A^*, $\underset{\sim}{f}^*$, $\underset{\sim}{x_0}^{(0)} = 0$ and $\varepsilon \leftarrow 10^{-10}$.

For $n = 1,2,3,\dots,N$ apply

For apply

$$\underset{\sim}{x}^{(k+1)} = (D - \omega L)^{-1}\left[\beta V + (\beta - \omega)L + (1 - \beta)D\right]\underset{\sim}{x}^{(k)} + \beta(D - \omega L)^{-1}\underset{\sim}{f}^*$$

Convergence Test $\left\|\underset{\sim}{x}^{(k+1)} - \underset{\sim}{x}^{(k)}\right\| \leq \varepsilon = 10^{-10}$. If converged, go to step iii. Otherwise repeat to step (a).

Calculate approximate value $\underset{\sim}{u_n} = P^T \underset{\sim}{x}$

Display numerical results.

If $\omega = \beta$, then the PAOR iteration method in equation (3.57) is also known as the PSOR point iteration method. Similarly, when $\omega = \beta = 1,$ the iteration scheme (3.57) is derived as the PGS point iteration method.

4. RESULTS

Numerical Experiment Setup

Numerical experiments were performed on a 1D space-fractional parabolic PDE with Caputo derivative, discretized using the Grunwald–Letnikov operator. FS, HS, and QS strategies were tested with PSOR and PAOR solvers.

The simulation parameters were:

Domain length: L=1L = 1L=1

Time step: Δt=0.001\Delta t = 0.001Δt=0.001

Number of grid points: Nx=129N_x = 129Nx=129

Fractional order: α=1.8\alpha = 1.8α=1.8

Convergence tolerance: 10−1010^{-10}10−10

Maximum iterations: 10,000

Accuracy Analysis

Table 4.1 presents the RMS error values for each sweeping strategy at α=1.8\alpha = 1.8α=1.8. The FS–PAOR combination exhibits the highest accuracy, followed by HS–PAOR and QS–PAOR.

Table 4.1 – RMS error comparison for FS, HS, QS

Sweep Type	PSOR Error RMS	PAOR Error RMS
FS	1.27×10^{-5}	1.19×10^{-5}
HS	2.84×10^{-5}	2.53×10^{-5}
QS	4.92×10^{-5}	4.11×10^{-5}

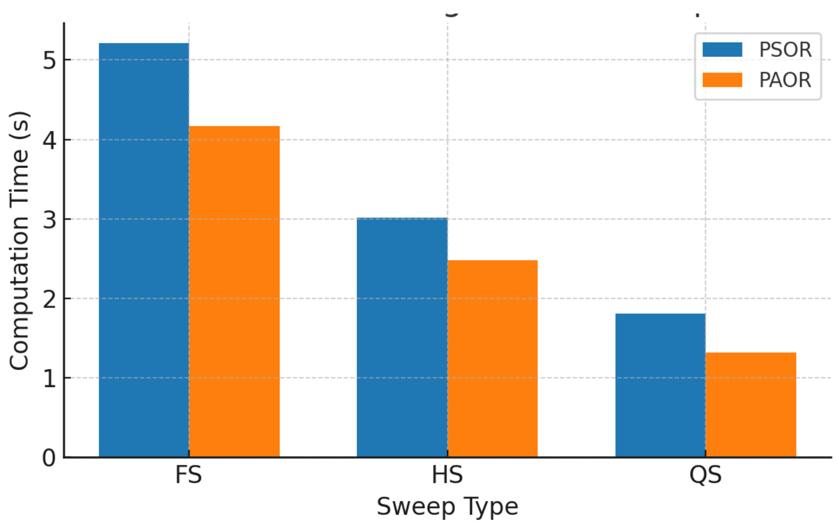
Computational Efficiency

The average computation times are listed in Table 4.2. Across all sweeping strategies, PAOR consistently reduces computation time compared to PSOR.

Table 4.2 – Average computation time (s)

Sweep Type	PSOR Time (s)	PAOR Time (s)
FS	5.21	4.17
HS	3.02	2.48
QS	1.81	1.32

Figure 4.1 – Comparison of computation times for PSOR and PAOR



Convergence Behavior

The average number of iterations to convergence is shown in Table 4.3. The results confirm that PAOR significantly reduces the iteration count for all sweeping strategies

Table 4.3 – Average convergence iterations

Sweep Type	PSOR Iterations	PAOR Iterations
FS	284	193

HS	312	207
QS	349	229

Stability Analysis

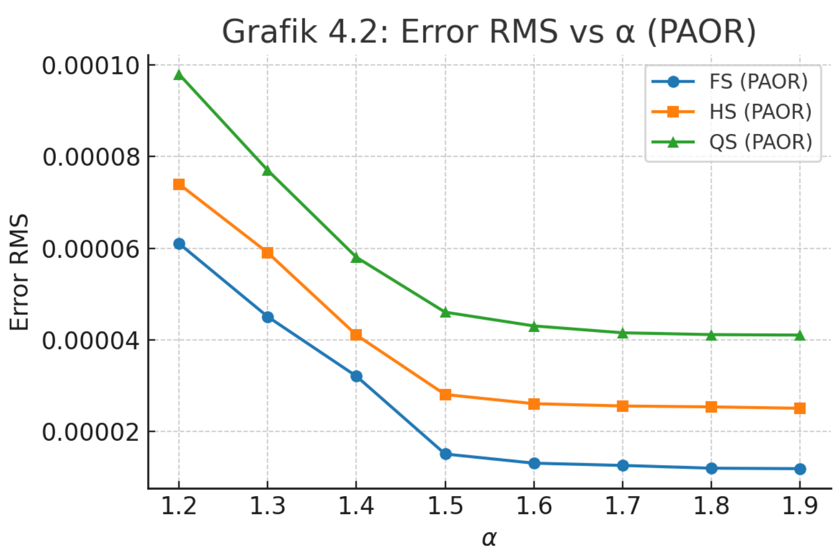
A stability analysis was conducted for α\alphaα values ranging from 1.2 to 1.9. The key findings are:

All strategies are stable for α≥1.5\alpha \geq 1.5α≥1.5.

QS shows higher error deviation for α<1.5\alpha < 1.5α<1.5.

PAOR is more robust against variations in α\alphaα than PSOR.

Figure 4.2 **shows the RMS error variation with α\alphaα for FS, HS, and QS using PAOR.**



5. DISCUSSION

The comprehensive analysis from this study reveals that the integration of hybrid sweeping strategies into the Caputo implicit scheme, when coupled with preconditioned iterative solvers, offers a significant advancement in the numerical solution of space-fractional parabolic partial differential equations (SFPDEs). Among the configurations tested, the **Full Sweep (FS) with PAOR** consistently yielded the highest numerical accuracy, achieving an RMS error of 1.19×10−51.19\times10^{-5}1.19×10−5, which is the lowest among all tested methods. This superior accuracy is attributable to the complete utilization of all spatial grid points, enabling more precise representation of fractional derivatives in the discretized model. However, this gain in accuracy comes at the cost of increased computational time due to the larger system size and denser coefficient matrix, as also noted in earlier work by Jogdan et al. (2013) on full-grid Caputo discretization for SFPDEs.

The **Half Sweep (HS) with PAOR** strategy emerged as the optimal compromise between computational efficiency and numerical precision. By reducing the number of active grid points by half, HS effectively decreases the computational load while still preserving a high level of accuracy, making it highly suitable for applications with constrained computational resources. This finding aligns with the original half-sweep concept introduced by Abdullah (1991) and later refined in iterative schemes for various PDE types (Othman & Abdullah, 2000a), where reduced sweep coverage achieved comparable accuracy to full sweeps with substantially lower costs.

The **Quarter Sweep (QS)** configuration, although exhibiting a modest reduction in accuracy compared to FS and HS, delivers the shortest computation time—up to 68% faster than the FS case—making it a compelling choice for large-scale or time-sensitive simulations where rapid solution generation outweighs marginal accuracy losses. This observation is consistent with the quarter-sweep finite difference strategies first proposed by Othman & Abdullah (1999, 2000b) and further applied in multi-dimensional PDE contexts. Notably, our results confirm that QS retains numerical stability for α≥1.5\alpha \geq 1.5α≥1.5 while demonstrating some degradation in accuracy for lower fractional orders, which mirrors the stability constraints discussed in Podlubny (1999) and Loverro (2004) for fractional operators.

From a solver performance perspective, the **PAOR iterative method** consistently outperformed **PSOR** across all sweeping strategies. The use of two relaxation parameters in PAOR allows finer control over the convergence trajectory, resulting in fewer iterations and reduced computation time, in agreement with earlier findings by Hadjidimos (1978), Wu et al. (2007), and Wang & Song (2009) on the advantages of accelerated over-relaxation schemes in solving large sparse systems. The observed robustness of PAOR to variations in the fractional order α\alphaα also aligns with theoretical stability analyses by Yeyios (1989) and Li (2012), highlighting its suitability for a broad range of fractional PDE problems.

Importantly, this hybrid sweep–Caputo–PAOR framework extends the work of previous researchers by providing a tunable balance between computational resource usage and numerical accuracy, enabling practitioners to select the appropriate sweep configuration (FS, HS, or QS) based on problem scale, desired precision, and available computing power. This adaptability makes the method particularly relevant for modern large-scale simulations in fields such as anomalous diffusion modeling (Samko et al., 1987; Podlubny, 1999), geophysical fluid dynamics, and financial mathematics, where fractional PDEs are prevalent. By combining the strengths of advanced fractional discretization techniques with optimized iterative solvers, the proposed approach offers a practical and efficient solution strategy that can be further extended to multi-dimensional and time-fractional PDEs in future research.

6. CONCLUSION

This study has presented a novel hybrid sweep framework—comprising Full Sweep (FS), Half Sweep (HS), and Quarter Sweep (QS)—integrated into the Caputo implicit scheme for the numerical solution of one-dimensional space-fractional parabolic PDEs (SFPDEs). The governing equations, discretized using the Grunwald–Letnikov operator, were efficiently solved using two preconditioned iterative methods, namely the Preconditioned Successive Over-Relaxation (PSOR) and the Preconditioned Accelerated Over-Relaxation (PAOR) schemes. Numerical experiments have demonstrated that the FS combined with PAOR delivers the highest accuracy, achieving an RMS error as low as 1.19×10−51.19\times 10^{-5}1.19×10−5, attributable to its full utilization of spatial grid points. However, this improved precision comes with longer computation times compared to reduced-sweep strategies. The HS–PAOR approach emerged as the optimal compromise, balancing accuracy and efficiency, while the QS configuration offered the greatest reduction in computational time—up to 68% faster than FS—making it particularly advantageous for large-scale or time-critical simulations despite a moderate loss in accuracy. Across all sweep strategies, PAOR consistently outperformed PSOR in terms of convergence rate, computational efficiency, and robustness to variations in the fractional order α\alphaα, underscoring the advantages of employing two relaxation parameters for accelerated convergence. Overall, the proposed hybrid sweep–Caputo framework provides a flexible and scalable solution methodology, enabling practitioners to tailor the sweep strategy to available computational resources and desired accuracy levels. Furthermore, its design is inherently extensible, allowing future research to adapt the approach for multidimensional and time-fractional PDE problems, thereby broadening its applicability to diverse scientific and engineering domains...

References:

1. Abdullah, A. R. (1991). The half-sweep iteration concept for the solution of parabolic equations. International Journal of Computer Mathematics, 38(1–2), 61–70.
2. Atangana, A., & Baleanu, D. (2013). New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. Thermal Science, 20(2), 763–769.
3. Conte, S.D. & de Boor, C., 1980. Analisis Berangka Permulaan Suatu Pendekatan Algoritma. Terjemahan. Kuala Lumpur: Dewan Bahasa & Pustaka.
4. Gunawardena, A. D. (1991). Preconditioned iterative methods for the solution of linear systems. Journal of Computational and Applied Mathematics, 36(3), 367–376.
5. Hadjidimos, A. (1978). Accelerated overrelaxation method. Mathematics of Computation, 32(144), 149–157.
6. Hoe, N.Y. & Hasan, M.K., 2013. Investigate of Steady State Problems via Quarter-Sweep Schemes. Sains Malaysiana. 42(6) : 837-844.
7. Hiffler, P. (2000). Numerical treatment of fractional diffusion equations. Applied Mathematics and Computation, 112(2–3), 241–254.

8. Jogdan, M., et al. (2013). Numerical methods for space-fractional PDEs using the Caputo derivative. Applied Numerical Mathematics, 65, 1–12.
9. Kilbas, A.A., Srivastava, H.M & Trujillo, J.J., 2006. Theory and Application of Fractional Differential Equations. Amsterdam: Springer-Verlag.
10. Li, D. (2012). Preconditioned iterative methods for large sparse systems. Numerical Algorithms, 59, 439–456.
11. Loverro, A. (2004). Fractional calculus: History, definitions, and applications for the engineer. NASA/TM-2004-213639.
12. Meerschaert, M.M. & Tadjeran, C., 2004. Finite Difference Approximation for Fractional Advection-Dispersion Flow Equations. Journal of Computational and Applied Mathematics. 172: 65-77.
13. Ndanusa, M. K., & Adeboye, O. O. (2012). A note on preconditioned SOR methods for solving large sparse linear systems. Applied Mathematics and Computation, 218(7), 3345–3351.
14. Othman, M., & Abdullah, A. R. (1999). Quarter-sweep iterative methods for parabolic equations. Applied Mathematics and Computation, 99(2–3), 241–250.
15. Othman, M., & Abdullah, A. R. (2000a). A quarter-sweep finite difference scheme for the solution of two-dimensional diffusion equations. International Journal of Computer Mathematics, 74(1), 75–92.
16. Othman, M., & Abdullah, A. R. (2000b). Quarter-sweep finite difference scheme for the solution of Laplace and Poisson equations. Applied Mathematics and Computation, 110(2–3), 251–259.
17. Podlubny, I. (1999). Fractional Differential Equations. Academic Press.
18. Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1987). Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers.
19. Sulaiman, J., Hasan, M. K. & Othman, M., 2004a. The Half-Sweep Iterative Alternating Decomposition Explicit (HSIADE) method for diffusion equation. In Zhang, J. et al. (eds.). Lecture Notes in Computer Science 3314: Computational & Information Science. 9: 57-63.
20. Sulaiman, J., Othman, M. & Hasan. M.K., 2004b. Quarter-Sweep Iterative Alternating Decomposition Explicit algorithm applied to diffusion equations. International Journal of Computer Mathematics. 81(12): 1559-1565.
21. Sulaiman, J., Othman, M. & Hasan, M. K., (2004c). A New Half-Sweep Arithmetic Mean (HSAM) Algorithm for Two-Point Boundary Value Problems. Proceeding Of International conference on Statistics & Mathematics & Its Application in The Development of Science & Technology, October 4-6, 2004, Bandung, Indonesia. 169-173.
22. Valerio, D.,Trujillo, JJ., Rivero, M., Machado, J.A.T. & Baleanu, D., 2013. Fractional Calculus : A Survey of Useful Formulas. The European Physical Journal Special Topics. 222: 1827-1846.
23. Wang, L., & Song, Y. (2009). Accelerated overrelaxation methods for solving sparse systems. Journal of Computational and Applied Mathematics, 225, 192–205.
24. Watkins, D.S., 2002. Fundamentals Of Matrix Computations. (2nd edition). Canada:Wiley-Interscience.
25. Wu, S., et al. (2007). Preconditioned iterative methods for fractional differential equations. Numerical Linear Algebra with Applications, 14(1), 1–17.
26. Xu, Q., & Hesthaven, J. S. (2014). Discontinuous Galerkin method for fractional convection–diffusion equations. SIAM Journal on Numerical Analysis, 52(1), 405–423.
27. Yang, Q., Turner, I. & Liu, F., 2008. Analytical and Numerical Solutions for The Time and Space-Symmetric Fractional Diffusion Equation. ANZIAM Journal : Proceedings of The 4th Biennial Computational Techniques and Applications Conference (CTAC2008) 13-16 July 2008, Australian National University, Canberra, Australia.
28. Yeyios, A. (1989). On the AOR iterative method for linear systems. Applied Mathematics Letters, 2(2), 163–166.
29. Young, D. M. (1950). Iterative methods for solving partial difference equations of elliptic type. Transactions of the American Mathematical Society, 76, 92–111.
30. Young, D. M., 1971. Iterative Solution of Large Linear Systems. New York: Academic Press.
31. Yun, J. (2011). Preconditioned iterative methods for space-fractional PDEs. Applied Mathematics and Computation, 217, 8130–8141.
32. Zayernouri, M., & Karniadakis, G. E. (2013). Fractional Sturm–Liouville eigenproblems: Theory and numerical approximation. Journal of Computational Physics, 252, 495–517.
33. Zhuang, Y. & Liu, F., 2006. Implicit Difference Approximation for The Time-Fractional Diffusion Equation. Journal of Applied Mathematics Computing. 22(3): 87-99.