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Article

SOLE CLASS OF MULTIFOLD WEIGHT THREEFOLD LINEAR CODES

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Abstract: In this paper, the weight distributions of the multifold weight classes of threefold linear codes over GF (Q) are established and given. The ir-reducible cyclic codes are the source of the optimal codes in these codes. Additionally, they feature multiple codes that are optimal. AMS Subject Classification: 94B05, 68R01, 44A05. Key Words and Phrases: Cyclotomic Classes; Gaussian Periods; Lin-ear Codes; Trace Function; Irreducible Cyclic Codes
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1. INTRODUCTION

Let $Q = P^s$ and $\mu = Q^m$ in this paper, where P is a prime and m and s are positive integers. Let $l = (\mu \ 1) / N$. Let N ¿ 1 be a positive integer that divides P 1. A linear [l; k; d] code over Q is a k-dimensional subspace of Q^l with the lowest distance d. Let N be a primitive element of $GF(\mu)$. The set

$$C=Tr(\gamma), Tr(\gamma\theta), \ldots \in Tr(\gamma\theta)^{l-1} : \gamma \quad GF(\mu) \quad (1.1)$$

It is known as an irreducible cyclic [l, m_0] code over GFQ when m_0 divides. The tracing function from GF (i) onto GFQ is represented by the symbols m and Tr.The weight distribution of irreducible cyclic codes is usually somewhat com-plicated [13].In certain special circumstances, the weight distribution is known. the following list of cases.

- The codes have two weights when $N/(Q^i + 1)$ for some i being a divisor of m/2. This is known as the semi positive case. Goethalr and Delscerete examined these codes [5]. McEliece [15], and McEliece and Baumert [2].
- $Ord_Q(N) = (N \ 1)$, and N is a prime with $N = 3(mod\ 4)$. The weight distribution was established by Mykkeltveit and Baumert [2].
- $Q \equiv 1(mod N)$ and gcd(m, N = 1), the weight distributions is known [6].
- N is even, gcd (n, N = 1), $Q \ 1 = N/2(mod N)$ and gcd ((r-1)/(Q-1) mod N, N)=2, the weight distribution is known [6].
- N = 2, the weight distribution was found by Baumert and McEliece [1].
- When N=3 and N=4, the distribution of weights is determined [6]. You may find the irreducible cyclic codes in [5, 9, 11, 12, 16, 18, 20, 21, 24].

Let

$$D = \{d_1, d_2, \ldots, d_n\} \subseteq GF(\mu)^*.$$

Define a linear code of length n over $GF(q)$ by letting D be the defining set.

$$C = \{ \mathrm{Tr}(yd_1), \mathrm{Tr}(yd_2), \ldots, \mathrm{Tr}(yd_n) \ : y \in GF(\mu) \}. \quad (1.2)$$

Since every linear code over GF (Q) may be formed in this manner, it is in fact a generic construction of linear codes. Depending on whatever defining set D is chosen, it can generate either excellent or terrible code.

The irreducible cyclic code of [1] is punctured from C_D up to coordinator permutations when D is a subset of the subgroups generated by β^N, where β is a generator of $GF(\mu)^*$.

The codes C_D punctured from the irreducible cyclic code in the semi-primitive case were addressed in [8], [7], and [23]. In addition to presenting a class of two weight codes, a class of three weight codes, and a class of four weight codes, the purpose of this work is to investigate the punctured code C_D in two more non-semi-primitive situations. Optimal codes are included in the two groups of codes discussed in this study.

2. PRELIMINARIES

The introduction of the multifold weight codes requires cyclotomic classes and Gaussian periods. We also give some supplementary results that are required for the sequel.

Assume that a primitive element of $GF(\mu)$ is B. For j=0,1,..., $N - i$, de-fine $C^{(N,\mu)} = B^+(B^N)$, where (B^N) denotes the subgroup of $GF(\mu)^*$ created by

$B^N A$. In $GF(\mu)$, the cyclotonic classes of order N are represented by $C_j^{(N,\mu)}.A$.

The definition of the order N cyclotomic numbers is

$$(j, l)^{N,\mu} = [(C_j^{N,\mu} + 1) \cap C_l^{N,\mu}]$$

for all $0 \leq j \leq N-1$ [22]. The Gaussian period are defined by

$$\eta_j^{N,\mu} = \sum_{y \in C_j^{N,\mu}} \chi(y), \quad i = 0, 1, \ldots, N-1$$

where the canonical additive character of $GF(\mu)$ is χ. We must ascertain the values of the Gaussian periods for our future applications. Although it is often highly difficult, there are some unique situations in which it is possible.

The definition of the period polynomials $\psi(N, \mu)(Y)$ is

$$\psi_{(N,\mu)}(Y) = \prod_{j=0}^{N-1} \left(Y - \eta_j^{(N,\mu)} \right)$$

According to [14], $\psi_{(N,\mu)}(Y)$ is a polynomial with integer coefficients. The proofs of the following six lemmas are available in [14].

3. The Weight formula for the code C_D

For every $y \in GF(\mu)$, it is convenient to define the weights in the code C_D of (1.2).

$$C_y = \{(Tr(yd_1), Tr(yd_2), \ldots, Tr(yd_n)\}$$

The Hamming weight $W(C_y)$ of C_y is $n - N_y(0)$, where

$$N_y(0) = |\{1 \le i \le n : Tr(yd_j) = 0\}|$$

for each $y \in GF(\mu)$.

It is easily seen that for any $D = \{d_1, d_2, \ldots, d_n\} \le GF(\mu)$. We have

$$QN_y(0) = \sum_{j=1}^{n} \sum_{\chi \in GF(Q^*)} B(\chi Tr(yd_j))$$

$$= n + \sum_{j=1}^{n} \sum_{\chi \in GF(Q)} \chi(xyd_j) \qquad (3.1)$$

where χ represents the canonical additive character of $GF(\mu)$ and B represents the canonical additive character of $GF(Q)$.
The dimension of C_D for arbitary D $\subseteq GF(\mu)$ relies on the choice of D and can be found as follows [8].

4. The Class of Two Wight Codes

Lemma 4.1. *Let* $N = 2^3$*, and define* C_1 *and* d_1 *as* $\mu = C^2+4d^2$, $C \equiv 1 (mod\ 2^3)$ *and* $P \equiv 1 (mod\ 2^3)$*, gcd(C, P)=1.TheseconstraintsdetermineC*$_1$ *and* d_1 *uniquely up to sign.*

$$\psi_{2^3,\mu}(\chi) = \chi^2 - C\chi + d^2$$

Theorem 4.1. *Let* $P \equiv 1 \pmod{2^3}$ *and* $m \equiv 0 \pmod{2^3}$*. Let* C_1 *and* d_1 *be given by* $Q^{-12} = C^2 + 4d^2$*, with* $C_1 \equiv 1 \pmod{2^3}$*. and* $gcd(C_1, P) = 1$*. The code with the following weight distributive is then the set* C_D*.*

Weight	Frequency
0	1
$\frac{\mu-1}{2} - c$	$\frac{\mu-1}{2} - d$
$\frac{\mu-1}{2} + c$	$\frac{\mu-1}{2} + d$

The dual code $C_D^\perp$ *of the code* C_D *has parameters*

$$\frac{\mu-1}{2(Q-1)}, \quad \frac{\mu-1}{2(Q-1)} - m, \quad d^\perp, \qquad d^\perp \ge 3$$

Proof. From (3), we have

$$Q\,N_y(0) - n = \sum_{j=1}^{n} \sum_{\chi \in GF(Q^*)} \chi(xyd_j) = \eta_i^{(N,\mu)}$$

whenever $y \in C_i^{(N,\mu)}$. Hence

$$W(C_y) = n - N_y(0)$$

The weight distribution from lemma 4.1 follows. m is the code's dimension. $d_1 \neq 0$ since $gcd(C_1, P) = 1$. Consequently, $C^2 < \sqrt{\mu}$. As a result, C_D has a dimension of m, and $W(C_y) > 0$ for each nonzero $y \in GF(\mu)$. All we have to show is that $d^{\perp} \geq 3$. *Any point of distinct* d_j and d_i is linearly independent over $GF(g)$ since they represent separate cosets of the factor group $C^{(N,\mu)}/GF(Q^*)$..

Example 4.1 Let Q = 17 and m = 2. The C_D is a [8,2,$\bar{7}$] code over $GF(17)$ with the following weight distribution $1.16x^2 + 10x^9$

5. The class of Three Weight Distribution

Lemma 5.1. *Let* $N = 2^3$. *Let c and d be defined by* $4\mu = C^2 + 27d^2$, $C \equiv 1 \pmod 8$, *and if* $P \equiv 1 \pmod 8$, *then* $gcd(C, P) = 1$. *These limitations uniquely identify* C *and d up to sign, therefore we have*

$$\chi_{2^3,\mu} = x^3 - \frac{c}{2}x^2 + \left(3\mu - \frac{9}{4}\right)x - c\mu - \frac{27d}{8}$$

Assume that $P \equiv 1 \pmod 8$, $m \equiv 0 \pmod 8$, and $N = 2^3$. Next, we get

$$\frac{r-1}{g-1} \mod N \equiv m \mod 2^3 \equiv 0.$$

indent Observe that any element of $GF(Q^*)$ may be written as $\gamma^j(\mu-1)/(Q-1)$, where γ is a fixed primitive element of $GF(\mu)$. Thus, $GF(Q)^* \subseteq C_0^{(N,\mu)}$

Theorem 5.1. *Let* $P \equiv 1 \pmod{2^3}$ *and* $m \equiv 0 \pmod{2^3}$. *Let* C *and d be given by*

$$4Q^{m/3} = C^2 + 27d^2, \quad C \equiv 1 \pmod{2^3},$$

and $\gcd(C, P) = 1$). *The code having the following weight distribution is then represented by the code*"

Weight	*Frequency*
0	*1*
$\frac{c-3-9d}{8}$	$\frac{\mu-1}{3}$
$\frac{c-3+9d}{8}$	$\frac{\mu-1}{3}$
$\frac{c+6}{8}$	$\frac{\mu-1}{3}$

The dual code $C_D^{\perp}$ of the code C_D has parameters

$$\frac{\mu-1}{3(Q-1)} \quad \frac{r-1}{3(Q-1)} - m\; d^{\perp}$$

where $d^{\perp} \geq 3$.

Proof. Since $d_1, d_2, \ldots, d_n$ from a complete set of coset representation of the factor group $C_0^{(N,\mu)}/GF(Q)^*$, we have

$$\cup_{x\in GF(Q)^*} \{xd_1, xd_2, .., xd_n\} = C_0^{N,\mu} \; By(3)$$

we have

$$Q\,N_y(0) - n = \sum_{j=1}^{n} \sum_{x\in GF(Q)^*} \chi(xyd_j) = \sum_{z\in C^{(N,\mu)}} \chi(xz) = z = \eta_i^{(N,\mu)}$$

whenever $y \in C_i^{(N,\mu)}$

The equation is $W(C_y) = n - N_y(0)$. from lemma 5.1.

The code's dimension C_D is denoted by m. $gcd(C, P) = 1$ yields $\mu - C^{y2} \neq 0$.

Notably, $4\mu^{y2^3} = C^2 + 27d^2$, $C \equiv 1 \pmod{2^3}$, and $gcd(C, P) = 1$. Therefore, for every nonzero $y \in GF(\mu)$, $W(C_y) > 0$ and C_D has dimension m. They to show that it is sufficient that $d^\perp \geq 3$. Since they represent separate cosets of the factor group $C_0^{(N,\mu)}/GF(Q)^*$, each pair of distinct d_j and d_i is linearly independent over $GF(Q)$.

Example 5.1 Let $Q = 19$ and $m = 3$, then C_D is a $[45, 3, 1010]$ code over $GF(19)$ using the weight distributions listed below $W(x) = 1 + 6x^{109/6} + 66x^{127/6} + 6x^{13/6}$

6. The class of Four Weight Distribution

Lemma 6.1. *Let* $N = 2^3$, *and let* C_2 *and* d_2 *be defined by* $\mu = C^2 + 4d^2$, *with* $C_2 \equiv 1 \pmod{2^3}$ *and* $P \equiv 1 \pmod{2^3}$. *Consequently,* $gcd(C_2, P) = 1$. *These limitations determine* d_2 *up to sign and* C_2 *uniquely..*

$$\psi_{2^3,\mu} = x^4 - 2c_2x^3 + (2\mu - 4d_2)x^2 + 4C_2\mu x - 4\mu^2$$

Theorem 6.1. *Think about* $P \equiv 1 \pmod{2^3}$ *and* $m \equiv 0 \pmod{2^3}$. *Assume that* $Q^{m/2} = C_2^2 + 4d_2^2$, $C_2 \equiv 1 \pmod{2^3}$ *provides* C_2 *and* d_2, *and that* $gcd(C_2, P) = 1$. *The code* C_D *has the weight distribution shown below. The dual code* $C_D^\perp$ *of the code* C_D *has parameters,*

$$\frac{\mu-1}{4(Q-1)}\;\frac{\mu-1}{4(Q-1)} - m\;d^\perp$$

where $d^\perp \geq 3$

Weight	Frequency
0	1
$\frac{\mu-1}{8} - \frac{C_2}{2} - d_2$	$\frac{\mu-1}{4} - d_2$
$\frac{\mu-1}{8} - \frac{C_2}{2} + d_2$	$\frac{\mu-1}{4} + d_2$
$\frac{\mu-1}{8} + \frac{C_2}{2} - d_2$	$\frac{\mu-1}{4} - d_2$
$\frac{\mu-1}{8} + \frac{C_2}{2} + d_2$	$\frac{\mu-1}{4} + d_2$

Proof. By (3) we have

$$Q\,N_y(0) - n = \sum_{j=1}^{n} \sum_{x \in GF(Q)^*} \chi(xyd_j) = \eta_L^{(N,\mu)}$$

where $y \in C_i^{(N^*,\mu)}$. Hence $W(C_y) = n - N_y(0)$.

from lemma 6.1.

Since $gcd(C_2, P) = 1$, $d_2 = 0$, the code has dimension m.

? The dimension of C_D is m, and for each nonzero y in $GF(\mu)$, $W(C_y) >$ 0. To show that $d^{\perp}$ 3 is sufficient, any pair of distinct d_j is linearly inde-pendent over $GF(Q)^*$ since they represent different cosets of the factor group $C_0^{(N,\mu)}/GF(Q)^*$.

Example 6.1 Let Q = 17 and m = 4. Then C_D is a [2,4,2] code over $GF(17)$ with the following weight distribution $1 + 2x^{-0.5} + 6x^{0.5} + 2x^{3.5} + 6x^{4.5}$

7. Conclusion

We infer that the optimality of secret sharing schemes is accelerated by imple-menting the single class of two weight, three weight, and four weight triple binary linear codes [22]..

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