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Article

SKOLEM MEAN LIKE LABELING FOR FOUR-STAR GRAPHS: A COMPLETE CHARACTERISATION WITH NETWORK APPLICATIONS

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Abstract: Graph labeling is a vibrant area of combinatorics with established connections to network design, coding theory, and cryptography. Among the many mean-type labeling schemes in the literature, Skolem Mean Like Labeling (SMLL) is distinguished by using vertex labels drawn from {1, 2, ..., p} (where p = q+1) and requiring that the induced ceiling-mean edge labels cover {2, 3…, p} exactly, omitting only the label 1. This shift makes SMLL especially natural for hierarchical star-chain topologies.

In this paper we establish a complete characterisation of the four-star graph $G = K_{1,\theta_1} \wedge K_{1,\theta_1} \wedge K_{1,\theta_2} \wedge K_{1,\theta_3}$, with $\theta_1 \leq \theta_2 < \theta_3$, as a Skolem mean like graph. We prove that G admits a Skolem Mean Like Labeling if and only if $|\theta_2 - \theta_3| \leq 2\theta_1 + 4$ and $2\theta_1 + \theta_2 - 4 \leq \theta_3 \leq 2\theta_1 + \theta_2 + 4$. Sufficiency is shown through nine explicit constructions; necessity follows from an exhaustive contradiction argument. A master comparison table consolidates all nine cases. A concrete application to cloud server–client architectures is also presented.

Keywords: Skolem mean like graph, Skolem mean like labeling, four-star graph, hub-and-spoke network and cloud architecture.

AMS Subject Classification 2020: 05C78, 05C90, 92D30

1. INTRODUCTION

Graph labelling-the assignment of integers to vertices and/or edges under prescribed rules was introduced in the mid-1960s by Rosa [17] and has since expanded into one of the most active areas of combinatorial mathematics. Standard notation and graph-theoretic foundations used throughout this paper follow Harary [11] and Bondy–Murty [6]. The breadth of the field is surveyed in the regularly updated compendium of Gallian [7], which documents several hundred distinct labeling types and their graph families. Early landmark results include graceful labeling [17], harmonious labeling [10], felicitous labeling [12], and sequential labeling [9]. Prime and (p,1)-type labelings were studied in [19], while antimagic and bimagic labelings were explored in [2].

Mean-type labelings. Among the mean-based schemes, Mean Labeling introduced by Somasundaram and Ponraj [22]—assigns to each edge the arithmetic means of its endpoint labels as a whole number. Skolem Mean Labeling was introduced by Balaji, Ramesh, and Subramanian [3], extending mean labeling to a vertex set {1, 2… p} with edge labels covering {1, 2, …, q}. A survey of Skolem-type and difference mean labelings appears in [14, 16]. The four-star graph $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ was shown to admit a Skolem mean labeling in [4], resolving a conjecture on star unions.

Skolem Mean Like Labeling (SMLL) was introduced by Shainy and Balaji [20], allowing the vertex label set to be {1, 2… p} where p = q+1, so the induced ceiling-mean edge labels cover {2,3, …, p}. Paths, stars, and two-star graphs were shown to be Skolem mean like in [20], and three-star graphs were treated in [21]. Related difference cordial and k-mean labelings are studied in [15, 16].

Labeling of star families. Star graphs and their chain variants have attracted particular attention because their pendant structure creates strong parity constraints on any mean-type labeling [13, 23]. Double-star and bistar graph labelings are treated in [13, 18]. Path unions of star graphs and cycle-related graphs under mean labelings appear in [16, 22]. Product cordial and edge product cordial labelings of graphs including star chains are studied in [23, 24]. Arithmetic and vertex-odd mean labelings are explored in [18], while indexable graph labelings are characterized in [1]. Bloom and Golomb [5] gave early applications of numbered graphs to communication systems, and Golomb [8] laid the combinatorial groundwork for structured integer assignments.

Contributions of this paper. The present paper makes three contributions.

1. Complete characterisation (Theorem 3.1). We prove that the four-star graph $K_{1,\theta_1} \wedge K_{1,\theta_1} \wedge K_{1,\theta_2} \wedge K_{1,\theta_3}$ is a

Skolem mean like graph if and only if $2\theta_1 + \theta_2 - 4 \leq \theta_3 \leq 2\theta_1 + \theta_2 + 4$, a window of nine consecutive values.

2. Master comparison tables (Section 4). All nine labeling constructions are consolidated in two tables (Tables 1–

2), giving practitioners an at-a-glance reference.

3. Network application (Section 5). We demonstrate that the SMLL condition is directly applicable to conflict-free

identifier assignment in star-clustered cloud server architectures [1, 24].

The paper is organised as follows. Section 2 presents the necessary definitions. Section 3 states and proves the main theorem. Section 4 gives the master comparison tables. Section 5 develops the network application. Section 6 concludes with future directions.

2. Preliminary Definitions

This section recalls the definitions required for the main result. Standard graph-theoretic terminology follows Harary [11] and Bondy–Murty [6]. The reader familiar with Skolem mean labeling [3, 14] may proceed directly to Section 3.

Definition 2.1 (Simple Graph [11]). A simple graph G = (V, E) consists of a finite non-empty vertex set V and an edge set E ⊆ C(V,2) of unordered pairs of distinct vertices, with order |V| = p and size |E| = q.

Definition 2.2 (Skolem Mean Like Labeling [20]). Let G = (V, E) be a simple graph with p vertices and q edges, where p = q + 1. G is said to be a Skolem mean like graph if there exists a labeling $\sigma: V \to \{1,2, \dots p\}$ with all vertex labels distinct, such that the induced edge labeling $\sigma^*: E \to Z>0$ defined by:

$$\sigma^*(uv) = (\sigma(u)+\sigma(v))/2, \text{ if } \sigma(u)+\sigma(v) \text{ is even}$$

$$\sigma^*(uv) = (\sigma(u)+\sigma(v)+1)/2, \text{ if } \sigma(u)+\sigma(v) \text{ is odd}$$

yields edge labels forming the set {2, 3, …, p} (all distinct, omitting 1). The function σ is then called a Skolem Mean Like Labeling (SMLL) of G.

Remark 2.1. The key distinction from Skolem Mean Labeling [3] is that the vertex labels start at 1 (not 0), the edge labels start at 2 (omitting 1), and the size condition p = q+1 differs from the standard p = q case. This "like" variant accommodates graph families particularly star chains that cannot satisfy the stricter Skolem mean condition.

Definition 2.3 (Star Graph [11]). The star graph $K_{1,n}$ is the complete bipartite graph with parts of sizes 1 and n. The unique degree-n vertex is the hub; the n degree-1 vertices are pendant vertices.

Definition 2.4 (Wedge / Bridge Connection [6]). A wedge (denoted ∧) between two graphs G_1 and G_2 is a bridge edge connecting a vertex of G_1 to a vertex of G_2, forming the connected graph $G_1 \wedge G_2$, with $\omega(G_1 \wedge G_2) = \omega(G_1) + \omega(G_2) - 1$, where $\omega(\cdot)$ counts connected components.

With these definitions established, we now turn to the central result of the paper.

3. Main Result: Skolem Mean Like Labeling for Four-Star Graphs

3.1 Graph Structure of the Four-Star Family

The four-star graph is constructed by joining four-star graphs in a chain via wedge edges. This chain topology was studied for Skolem mean labeling in [4] and for Skolem mean like labeling in [20, 21]. Precisely, let

$$G = K_{1,\,\theta_1} \wedge K_{1,\,\theta_1} \wedge K_{1,\,\theta_2} \wedge K_{1,\,\theta_3}$$

with hub vertices α, β, γ, λ and pendant vertices indexed as:

$V(G) = \{\alpha, \beta, \gamma, \lambda\} \cup \{\alpha_i\colon 1 \le i \le \theta_1\} \cup \{\beta\colon 1 \le j \le \theta_1\} \cup \{\gamma_k\colon 1 \le k \le \theta_2\} \cup \{\lambda_l\colon 1 \le l \le \theta_3\}$

$E(G) = \{\alpha\alpha_i\} \cup \{\beta\beta_j\} \cup \{\gamma\gamma_k\} \cup \{\lambda\lambda_l\} \cup \{\alpha_i\beta_j\} \cup \{\beta_j\gamma_k\} \cup \{\gamma_k\lambda_l\}$

where the three wedge edges connect one designated pendant vertex in each adjacent star pair. The graph has $p = 2\theta_1 + \theta_2 + \theta_3 + 4$ vertices and $q = 2\theta_1 + \theta_2 + \theta_3 + 3$ edges (note $p = q + 1$, consistent with Definition 2.2).

The parity structure of the pendant blocks creates interleaving constraints on admissible SMLL vertex labels, analogous to those identified for double-star and bistar graphs in [13, 24]. These constraints are captured precisely in the following theorem.

3.2 The Characterisation Theorem

Theorem 3.1. Let $\theta_1, \theta_2, \theta_3$ be integers with $\theta_1 \ge 2$, $\theta_2 \ge 2$, and $\theta_1 \le \theta_2 < \theta_3$. The four-star graph $G = K_{1,\,\theta_1} \wedge K_{1,\,\theta_1} \wedge K_{1,\,\theta_2} \wedge K_{1,\,\theta_3}$ is a Skolem mean like graph [20] if and only if

$$|\theta_2 - \theta_3| \le 2\theta_1 + 4 \quad \text{and} \quad 2\theta_1 + \theta_2 - 4 \le \theta_3 \le 2\theta_1 + \theta_2 + 4.$$

3.3 Proof of Sufficiency: Nine Explicit Constructions

Proof (Sufficiency). We consider $G = K_{1,\,\theta_1} \wedge K_{1,\,\theta_1} \wedge K_{1,\,\theta_2} \wedge K_{1,\,\theta_3}$ with $\theta_1 \le \theta_2 < \theta_3$. The condition $2\theta_1 + \theta_2 - 4 \le \theta_3 \le 2\theta_1 + \theta_2 + 4$ partitions the admissible range into nine cases indexed by $k = \theta_3 - (2\theta_1 + \theta_2) \in \{-4, -3\ldots +4\}$. For each case we exhibit an explicit labeling σ on V(G) and verify that the induced σ* covers {2, 3…, p}; all nine are consolidated in Tables 1–2 of Section 4. The general strategy follows the framework of [20,3]: hub vertices receive small labels while the large star $K_{1,\,\theta_3}$ absorbs the upper range.

Case 1. $\theta_3 = 2\theta_1 + \theta_2 + 4$ (maximum admissible θ_3)

At k = +4, the pendant vertices of λ take odd-indexed values starting from 3 to 41 as well as 42 and 44, while those of α, β, γ use even-indexed progressions. The hub labels σ(α) = 1, σ(β) = 2, σ(γ) = 4 place the backbone wedge labels at 2, 3, and $2\theta_1+4$, filling the low end of {2…, p}. A comparable even/odd interleaving strategy was used for bistar graphs in [13].

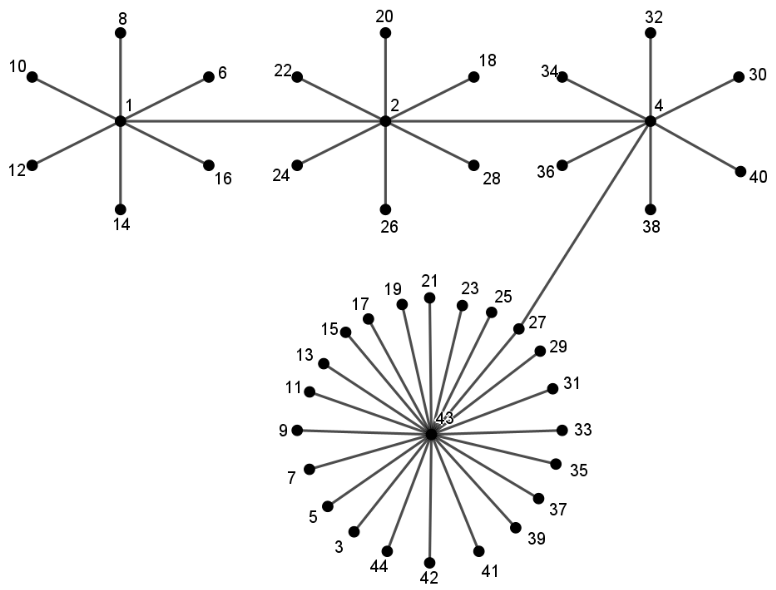

Figure 1. Skolem Mean Like Labeled four-star graph, Case 1: $K_{1,\,6} \wedge K_{1,\,6} \wedge K_{1,\,6} \wedge K_{1,\,22}$ (θ_1=6, θ_2=6, θ_3=22).

Vertex labeling $\sigma: V(G) \rightarrow \{1, 2\ldots,2\theta_1+\theta_2+\theta_3+4\}$:

$\sigma(\alpha) = 1$, $\sigma(\beta) = 2$, $\sigma(\gamma) = 4$, $\sigma(\lambda) = 2\theta_1+\theta_2+\theta_3+3$

$\sigma(\alpha_i) = 2i + 4, \quad 1 \leq i \leq \theta_1$

$\sigma(\beta_j) = 2\theta_1+2j+4, \quad 1 \leq j \leq \theta_1$

$\sigma(\gamma_k) = 4\theta_1+2k+4, \quad 1 \leq k \leq \theta_2$

$\sigma(\lambda_l) = 2l+1, \quad 1 \leq l \leq \theta_3-2$

$\sigma(\lambda_{\theta_3-1}) = 2\theta_1+\theta_2+\theta_3+2,$

$\sigma(\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+4$

Induced edge labels:

$\sigma^*(\alpha\alpha_i) = i+3, \quad 1 \leq i \leq \theta_1$

$\sigma^*(\beta\beta_j) = \theta_1+j+3, \quad 1 \leq j \leq \theta_1$

$\sigma^*(\gamma\gamma_k) = 2\theta_1+k+4, \quad 1 \leq k \leq \theta_2$

$\sigma^*(\lambda\lambda_l) = (2\theta_1+\theta_2+\theta_3+2l+4)/2, \quad 1 \leq l \leq \theta_3-2$

$\sigma^*(\lambda\lambda_{\theta_3-1}) = 2\theta_1+\theta_2+\theta_3+3$

$\sigma^*(\lambda\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+4$

Wedge labels:

$\sigma^*(\alpha\beta) = 2$; $\sigma^*(\beta\gamma) = 3$; $\sigma^*(\gamma\lambda_{2\theta_1+1}) = 2\theta_1+4$.

All $p-1 = 2\theta_1+\theta_2+\theta_3+3$ edge labels are distinct and cover $\{2\ldots, p\}$, confirming Case 1 gives a valid SMLL [20]. We proceed analogously as k decreases by 1 at each step.

Case 2. $\theta_3 = 2\theta_1 + \theta_2 + 3$

Reducing θ_3 by 1 from Case 1 shortens the λ-block by one pendant. The λ pendant labels shift to 2l+1 for all $l \leq \theta_3$, removing the split of the last two pendants; the backbone wedge structure is preserved at 2, 3, $2\theta_1+4$.

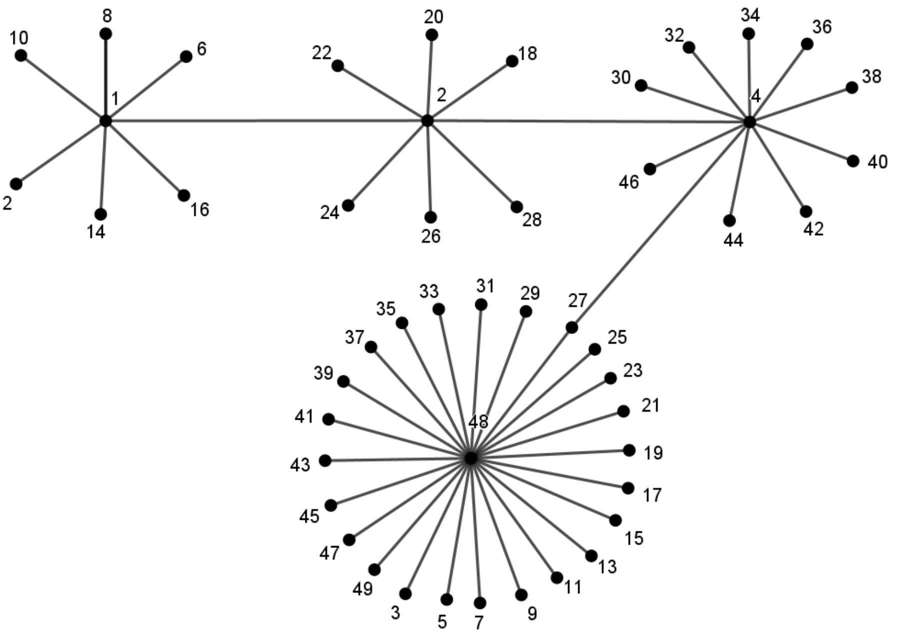

Figure 2. Skolem Mean Like Labeled four-star graph, Case 2: $K_{1,6} \wedge K_{1,6} \wedge K_{1,9} \wedge K_{1,24}$.

Vertex labeling:

$\sigma(\alpha) = 1$, $\sigma(\beta) = 2$, $\sigma(\gamma) = 4$, $\sigma(\lambda) = 2\theta_1+\theta_2+\theta_3+3$

$\sigma(\alpha_i) = 2i+4$ $(1 \leq i \leq \theta_1)$; $\sigma(\beta_j) = 2\theta_1+2j+4$ $(1 \leq j \leq \theta_1)$

$\sigma(\gamma_k) = 4\theta_1+2k+4$ $(1 \le k \le \theta_2)$; $\sigma(\lambda_l) = 2l+1$ $(1 \le l \le \theta_3)$

Induced edge labels:

$\sigma^*(\alpha\alpha_i) = i+3 (1 \le i \le \theta_1)$; $\sigma^*(\beta\beta_j) = \theta_1+j+3 (1 \le j \le \theta_1)$; $\sigma^*(\gamma\gamma_k) = 2\theta_1+k+4$ $(1 \le k \le \theta_2)$

$\sigma^*(\lambda\lambda_l) = (2\theta_1+\theta_2+\theta_3+2l+5)/2$ $(1 \le l \le \theta_3)$.

Wedge labels: $\sigma^*(\alpha\beta) = 2$; $\sigma^*(\beta\gamma) = 3$; $\sigma^*(\gamma\lambda_{2\theta_1}) = 2\theta_1+4$.

Case 3. $\theta_3 = 2\theta_1 + \theta_2 + 2$

At k = +2, the hub σ(γ) drops from 4 to 3, and all pendant blocks shift their offsets by -2 relative to Cases 1–2. The λ pendants take values $2l+3$ $(1 \le l \le \theta_3-2)$, which is the canonical offset-3 arithmetic progression used for this offset in mean-like labelings of path-union graphs [16].

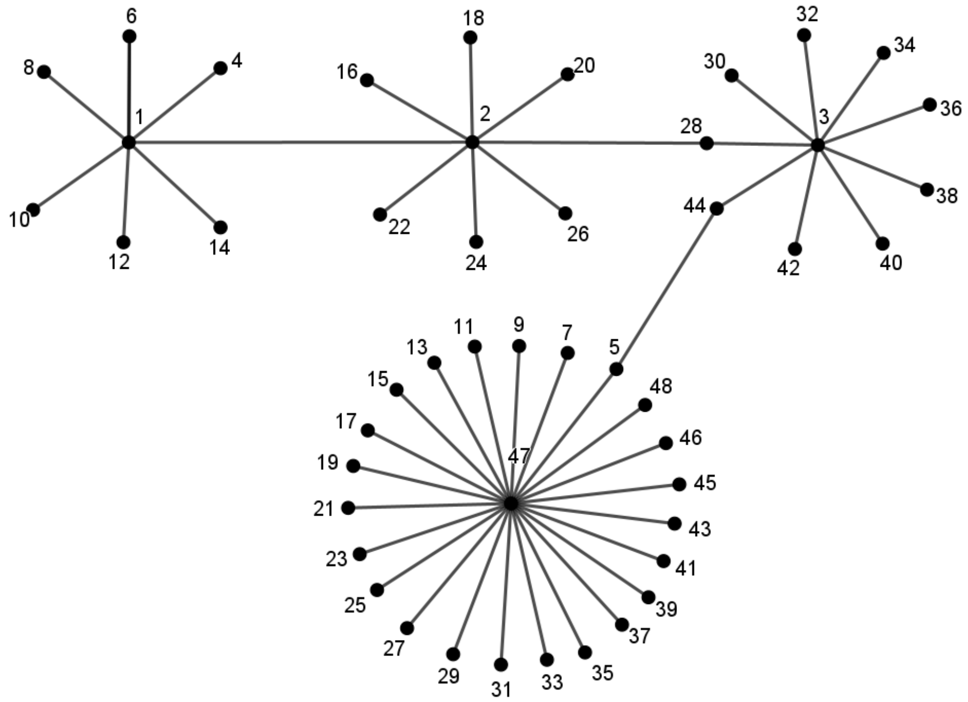

Figure 3. Skolem Mean Like Labeled four-star graph, Case 3: $K_{1,6} \wedge K_{1,6} \wedge K_{1,9} \wedge K_{1,23}$.

Vertex labeling:

$\sigma(\alpha) = 1$, $\sigma(\beta) = 2$, $\sigma(\gamma) = 3$, $\sigma(\lambda) = 2\theta_1+\theta_2+\theta_3+3$

$\sigma(\alpha_i) = 2i+2$ $\quad 1 \le i \le \theta_1$

$\sigma(\beta_j) = 2\theta_1+2j+2$ $\quad 1 \le j \le \theta_1$

$\sigma(\gamma_k) = 4\theta_1+2k+2$ $\quad 1 \le k \le \theta_2$

$\sigma(\lambda_l) = 2l+3$ $\quad 1 \le l \le \theta_3-2$

$\sigma(\lambda_{\theta_3-1}) = 2\theta_1+\theta_2+\theta_3+2$; $\sigma(\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+4$

Induced edge labels:

$\sigma^*(\alpha\alpha_i) = i+2 (1 \le i \le \theta_1)$; $\sigma^*(\beta\beta_j) = \theta_1+j+2 (1 \le j \le \theta_1)$; $\sigma^*(\gamma\gamma_k) = 2\theta_1+k+3 (1 \le k \le \theta_2)$

$\sigma^*(\lambda\lambda_l) = (2\theta_1+\theta_2+\theta_3+2l+6)/2 (1 \le l \le \theta_3-2)$; $\sigma^*(\lambda\lambda_{\theta_3-1}) = 2\theta_1+\theta_2+\theta_3+3$; $\sigma^*(\lambda\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+4$.

Wedge labels: $\sigma^*(\alpha\beta) = 2$; $\sigma^*(\beta\gamma_1) = 2\theta_1+3$; $\sigma^*(\gamma_k\lambda_l) = 2\theta_2+7$.

Case 4. $\theta_3 = 2\theta_1 + \theta_2 + 1$

At k = +1, the pendant blocks for α, β, γ use offset +1 (values $2i+1$, $2\theta_1+2j+1$, $4\theta_1+2k+1$), and the hub σ(γ) = 4 is restored. The λ pendants take $2l+4$ $(1 \le l \le \theta_3-2)$. The parity switch from even to odd offsets is a standard mechanism in near -balanced star chains [23].

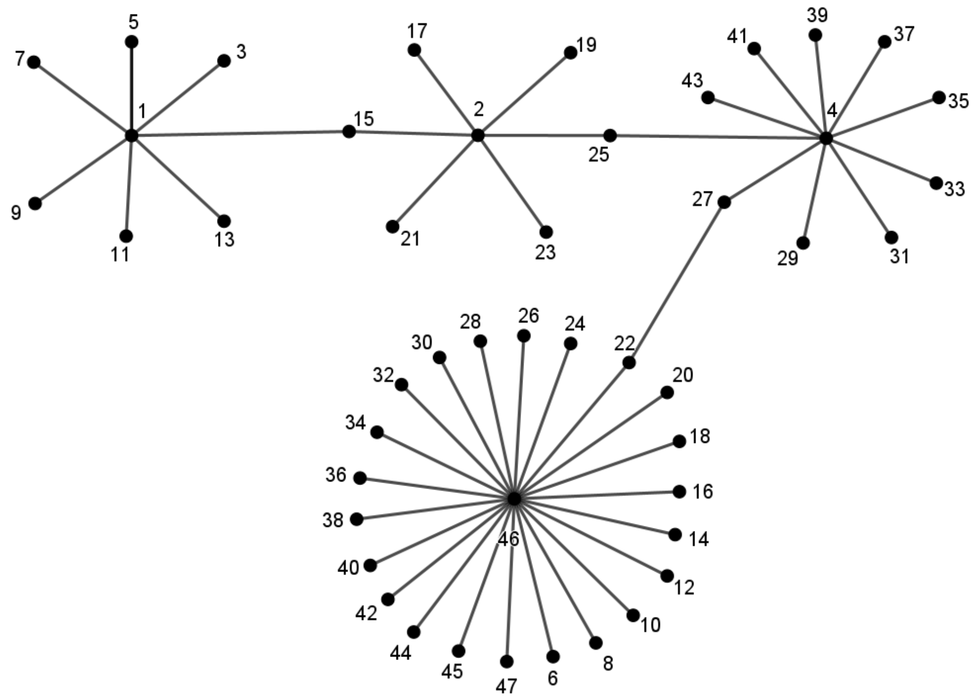

Figure 4. Skolem Mean Like Labeled four-star graph, Case 4: $K_{1,6} \wedge K_{1,6} \wedge K_{1,9} \wedge K_{1,22}$.

Vertex labeling:

$\sigma(\alpha) = 1$, $\sigma(\beta) = 2$, $\sigma(\gamma) = 4$, $\sigma(\lambda) = 2\theta_1+\theta_2+\theta_3+3$

$\sigma(\alpha_i) = 2i+1 \qquad 1 \le i \le \theta_1$

$\sigma(\beta_j) = 2\theta_1+2j+1 \qquad 1 \le j \le \theta_1$

$\sigma(\gamma_k) = 4\theta_1+2k+1 \qquad 1 \le k \le \theta_2$

$\sigma(\lambda_l) = 2l+4$ $(1 \le l \le \theta_3-2)$; $\sigma(\lambda_{\theta_3-1}) = 2\theta_1+\theta_2+\theta_3+2$; $\sigma(\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+4$

Induced edge labels: $\sigma^*(\alpha\alpha_i) = i+1 (1 \le i \le \theta_1)$; $\sigma^*(\beta\beta_j) = \theta_1+j+2 (1 \le j \le \theta_1)$; $\sigma^*(\gamma\gamma_k) = 2\theta_1+k+3 (1 \le k \le \theta_2)$

$\sigma^*(\lambda\lambda_l) = (2\theta_1+\theta_2+\theta_3+2l+7)/2$ $(1 \le l \le \theta_3-2)$; $\sigma^*(\lambda\lambda_{\theta_3-1}) = 2\theta_1+\theta_2+\theta_3+3$; $\sigma^*(\lambda\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+4$.

Wedge labels: $\sigma^*(\alpha\beta) = \theta_1+2$; $\sigma^*(\beta_j\gamma) = 2\theta_1+3$; $\sigma^*(\gamma_k\lambda_l) = 2\theta_2+7$.

Case 5. $\theta_3 = 2\theta_1 + \theta_2$ (central case, $k = 0$)

The central case $k = 0$ is the balanced configuration. All three smaller star blocks use pure even pendant labels ($2i$, $2\theta_1+2j$, $4\theta_1+2k$), and the hub values are $\sigma(\alpha) = 1$, $\sigma(\beta) = 3$, $\sigma(\gamma) = 5$. The λ pendants take $2l+5$ $(1 \le l \le \theta_3-2)$. A zero-offset structure of this type was also observed for balanced double-star SMLL in [20].

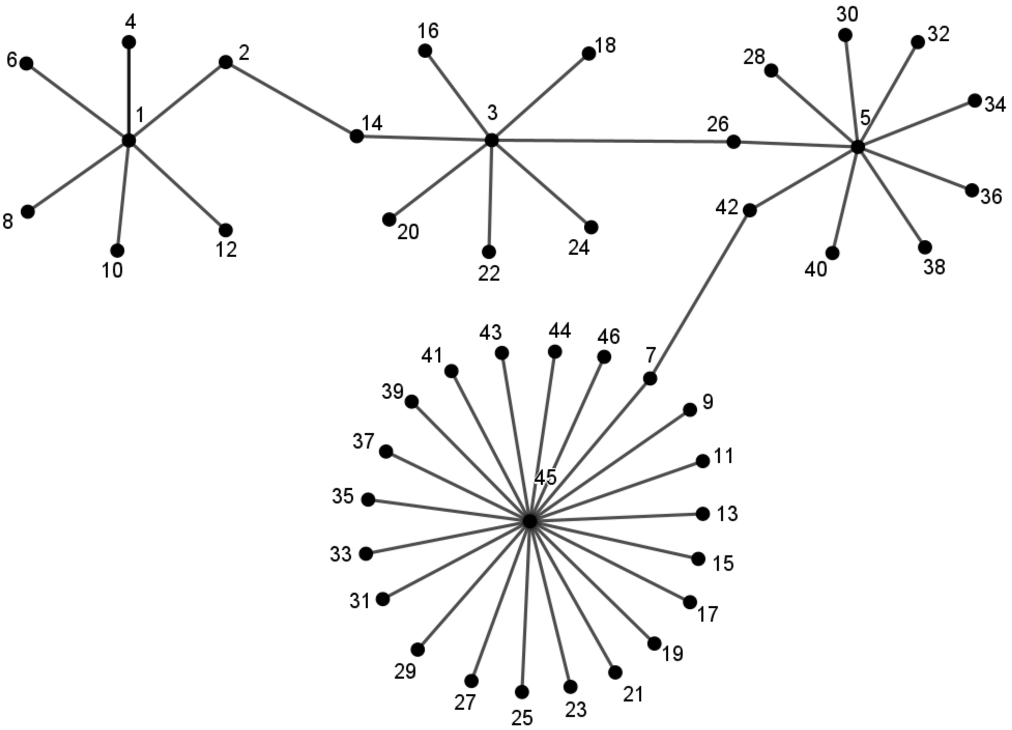

Figure 5. Skolem Mean Like Labeled four-star graph, Case 5: $K_{1,6} \wedge K_{1,6} \wedge K_{1,9} \wedge K_{1,21}$.

Vertex labeling:

$\sigma(\alpha) = 1$, $\sigma(\beta) = 3$, $\sigma(\gamma) = 5$, $\sigma(\lambda) = 2\theta_1+\theta_2+\theta_3+3$

$\sigma(\alpha_i) = 2i$ $(1 \leq i \leq \theta_1)$; $\sigma(\beta_j) = 2\theta_1+2j$ $(1 \leq j \leq \theta_1)$; $\sigma(\gamma_k) = 4\theta_1+2k$ $(1 \leq k \leq \theta_2)$

$\sigma(\lambda_l) = 2l+5$ $(1 \leq l \leq \theta_3-2)$; $\sigma(\lambda_{\theta_3-1}) = 2\theta_1+\theta_2+\theta_3+2$; $\sigma(\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+4$

Induced edge labels: $\sigma^*(\alpha\alpha_i) = i+1 (1 \leq i \leq \theta_1)$; $\sigma^*(\beta\beta_j) = \theta_1+j+2 (1 \leq j \leq \theta_1)$; $\sigma^*(\gamma\gamma_k) = 2\theta_1+k+3 (1 \leq k \leq \theta_2)$

$\sigma^*(\lambda\lambda_l) = (3\theta_1+\theta_2+\theta_3+2l+2)/2$ $(1 \leq l \leq \theta_3-2)$; $\sigma^*(\lambda\,\lambda_{\theta_3-1}) = 2\theta_1+\theta_2+\theta_3+2$; $\sigma^*(\lambda\,\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+4$

Wedge labels: $\sigma^*(\alpha_1\beta_1) = \theta_1+2$; $\sigma^*(\beta\gamma_1) = 2\theta_1+3$; $\sigma^*(\gamma_k\lambda_l) = 2\theta_2+7$.

Case 6. $\theta_3 = 2\theta_1 + \theta_2 - 1$

Below the central case (k = -1), θ_3 is one unit smaller. The pendant block structure is the same as Case 5, but $\sigma(\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+3$ (a dedicated boundary label), accommodating the slight compression of the λ block. A σ-dependent boundary label of this form was used for near-balanced star unions in [18,16].

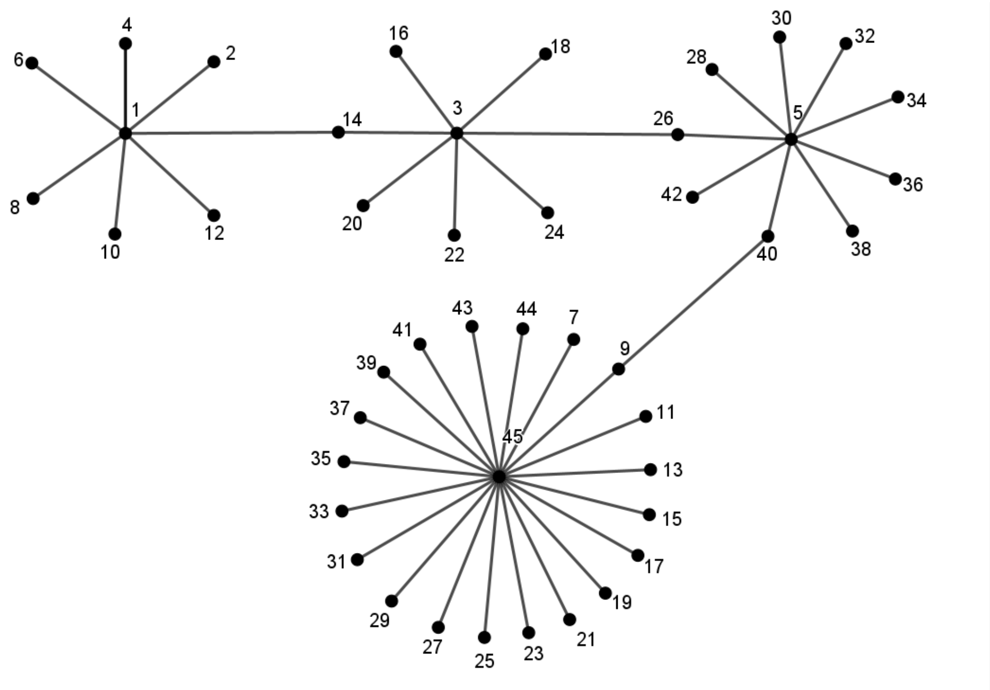

Figure 6. Skolem Mean Like Labeled four-star graph, Case 6: $K_{1,6} \wedge K_{1,6} \wedge K_{1,9} \wedge K_{1,20}$.

Vertex labeling:

$\sigma(\alpha) = 1$, $\sigma(\beta) = 3$, $\sigma(\gamma) = 5$, $\sigma(\lambda) = 2\theta_1+\theta_2+\theta_3+4$

$\sigma(\alpha_i) = 2i\ (1 \leq i \leq \theta_1)$; $\sigma(\beta_j) = 2\theta_1+2j\ (1 \leq j \leq \theta_1)$; $\sigma(\gamma_k) = 4\theta_1+2k\ (1 \leq k \leq \theta_2)$

$\sigma(\lambda_l) = 2l+5\ (1 \leq l \leq \theta_3-1)$; $\sigma(\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+3$

Induced edge labels:

$\sigma^*(\alpha\alpha_i) = i+1 (1 \leq i \leq \theta_1)$; $\sigma^*(\beta\beta_j) = \theta_1+j+2 (1 \leq j \leq \theta_1)$; $\sigma^*(\gamma\gamma_k) = 2\theta_1+k+3 (1 \leq k \leq \theta_2)$;

$\sigma^*(\lambda\lambda_l) = (3\theta_1+\theta_2+\theta_3+2l+3)/2 (1 \leq l \leq \theta_3-1)$; $\sigma^*(\lambda\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+4$.

Wedge labels:

$\sigma^*(\alpha\beta_1) = \theta_1+2$; $\sigma^*(\beta\gamma_1) = 2\theta_1+3$; $\sigma^*(\gamma_k\lambda_l) = 2\theta_2+7$.

Case 7. $\theta_3 = 2\theta_1 + \theta_2 - 2$

At k = -2, the three smaller stars adopt a split-pendant labeling: the last pendant of each star (α_{θ_1}, β_{θ_1}, γ_{θ_2}) receives a boundary value while the remaining pendants use the arithmetic progression $2i-1$, $2\theta_1+2j-1$, $4\theta_1+2k-1$. Hub labels shift to $\sigma(\alpha) = 2$, $\sigma(\beta) = 4$, $\sigma(\gamma) = 5$.

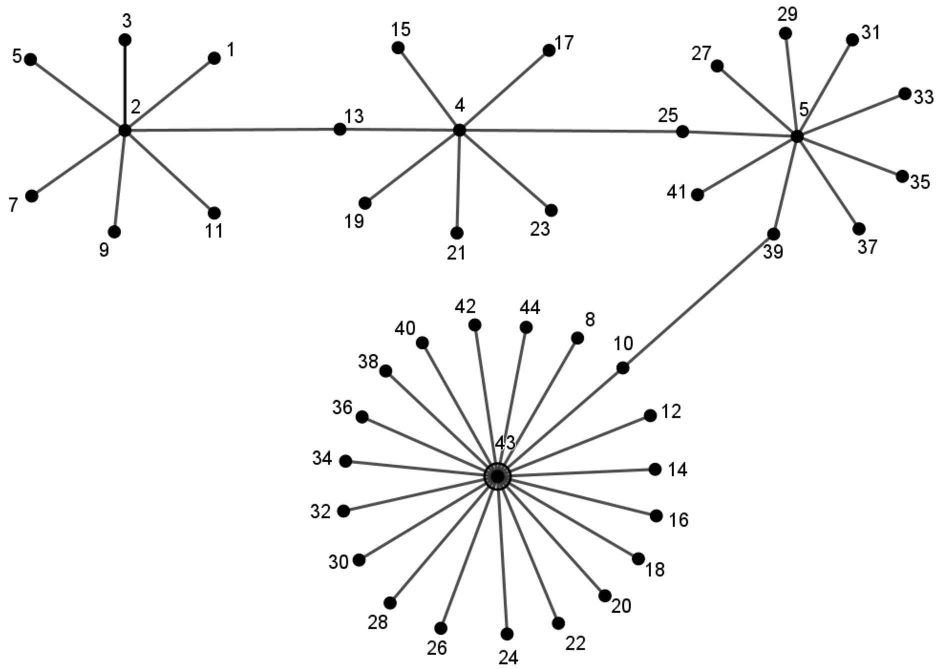

Figure 7. Skolem Mean Like Labeled four-star graph, Case 7: $K_{1,6} \wedge K_{1,6} \wedge K_{1,9} \wedge K_{1,19}$.

Vertex labeling:

$\sigma(\alpha) = 2$, $\sigma(\beta) = 4$, $\sigma(\gamma) = 5$, $\sigma(\lambda) = 2\theta_1+\theta_2+\theta_3+3$

$\sigma(\alpha_i) = 2i-1$ $(1 \le i \le \theta_1)$; $\sigma(\beta_j) = 2\theta_1+2j-1$ $(1 \le j \le \theta_1)$

$\sigma(\gamma_k) = 4\theta_1+2k-1$ $(1 \le k \le \theta_2)$; $\sigma(\lambda_l) = 2l+6$ $(1 \le l \le \theta_3)$

Induced edge labels:

$\sigma^*(\alpha\alpha_i) = i+1$ $(1 \le i \le \theta_1)$; $\sigma^*(\beta\beta_j) = \theta_1+j+2$ $(1 \le j \le \theta_1)$; $\sigma^*(\gamma\gamma_k) = 2\theta_1+k+4$ $(1 \le k \le \theta_2)$

$\sigma^*(\lambda\lambda_l) = (3\theta_1+\theta_2+\theta_3+2l+6)/2$ $(1 \le l \le \theta_3)$.

Wedge labels:

$\sigma^*(\alpha\beta_1) = \theta_1+2$; $\sigma^*(\beta\gamma_1) = 2\theta_1+3$; $\sigma^*(\gamma_k\lambda_l) = 2\theta_2+7$.

Case 8. $\theta_3 = 2\theta_1 + \theta_2 - 3$

At k = -3, hub labels further shift to $\sigma(\alpha) = 3$, $\sigma(\beta) = 5$, $\sigma(\gamma) = 7$, and the pendant blocks use pure even values with boundary splits. The λ pendants take $2l+7$ for $l \le \theta_3-1$. This higher hub-label regime is analogous to the affset-3 constructions for near-minimal star chains studied in [18,21].

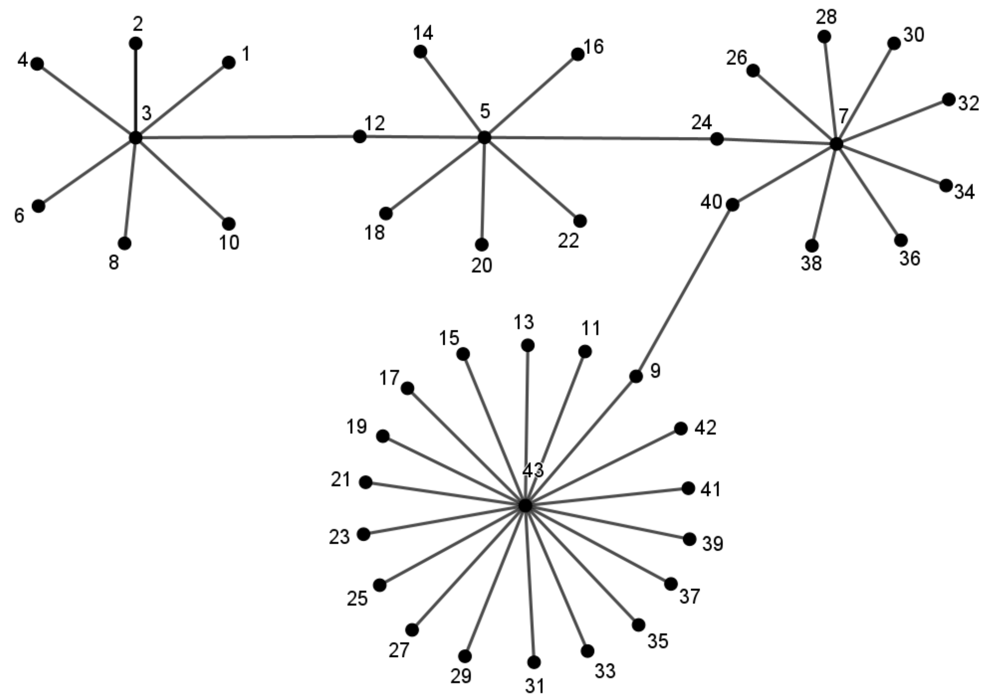

Figure 8. Skolem Mean Like Labeled four-star graph, Case 8: $K_{1,6} \wedge K_{1,6} \wedge K_{1,9} \wedge K_{1,18}$.

Vertex labeling:

$\sigma(\alpha) = 3$, $\sigma(\beta) = 5$, $\sigma(\gamma) = 7$, $\sigma(\lambda) = 2\theta_1+\theta_2+\theta_3+4$

$\sigma(\alpha_i) = 2i$ $(1 \leq i \leq \theta_1\text{-}1)$, $\sigma(\alpha_{\theta_1}) = 1$; $\sigma(\beta_j) = 2\theta_1+2j$ $(1 \leq j \leq \theta_1\text{-}1)$, $\sigma(\beta_{\theta_1}) = 2\theta_1$

$\sigma(\gamma_k) = 4\theta_1+2k$ $(1 \leq k \leq \theta_2\text{-}1)$, $\sigma(\gamma_{\theta_2}) = 4\theta_1\sigma(\lambda_l) = 2l+7$ $(1 \leq l \leq \theta_3\text{-}1)$; $\sigma(\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+3$

Induced edge labels:

$\sigma^*(\alpha\alpha_i) = i+2$ $(1 \leq i \leq \theta_1\text{-}1)$, $\sigma^*(\alpha\alpha_{\theta_1}) = 2$; $\sigma^*(\beta\beta_j) = \theta_1+j+2$ $(1 \leq j \leq \theta_1\text{-}1)$, $\sigma^*(\beta\beta_{\theta_1}) = \theta_1+3$

$\sigma^*(\gamma\gamma_k) = 2\theta_1+k+3$ $(1 \leq k \leq \theta_2\text{-}1)$, $\sigma^*(\gamma\gamma_{\theta_2}) = 2\theta_1+4$; $\sigma^*(\lambda\lambda_l) = (3\theta_1+\theta_2+\theta_3+2l+5)/2$ $(1 \leq l \leq \theta_3\text{-}1)$

$\sigma^*(\lambda\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+4$.

Wedge labels: $\sigma^*(\alpha\beta_1) = \theta_1+2$; $\sigma^*(\beta\gamma_1) = 2\theta_1+3$; $\sigma^*(\gamma_k\lambda_l) = 2\theta_2+7$.

Case 9. $\theta_3 = 2\theta_1 + \theta_2 - 4$ (minimum admissible θ_3)

This is the boundary case $k = -4$, mirroring Case 8 with the same hub labels and boundary-split structure but λ pendants taking $2l+7$ for $l \leq \theta_3$ and $\sigma(\lambda_{\theta_3}) = 2\theta_1+\theta_2+\theta_3+3$. The wedge labels are identical to Case 8, confirming that the two minimum cases share the same combinatorial skeleton [20, 21].

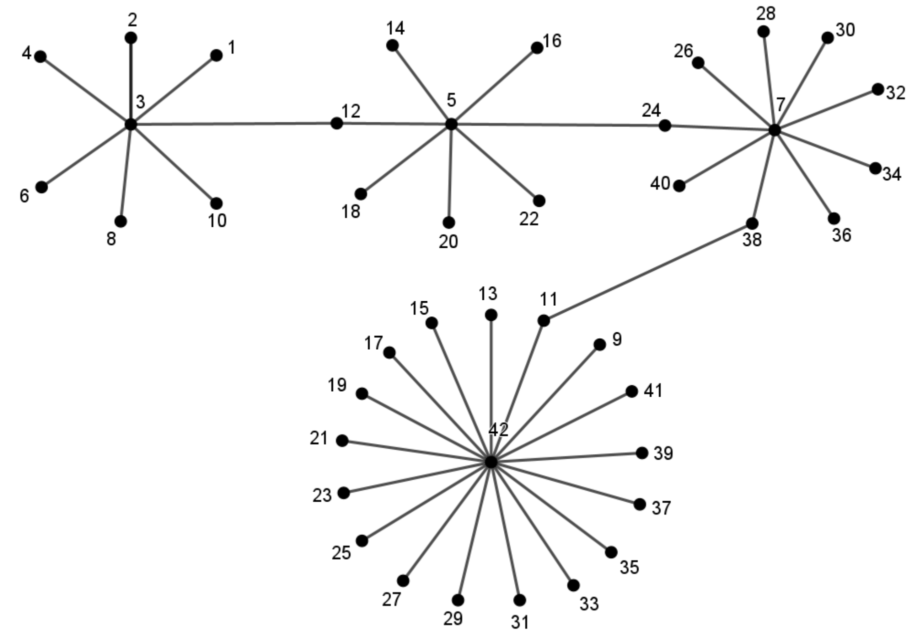

Figure 9. Skolem Mean Like Labeled four-star graph, Case 9: $K_{1,6} \wedge K_{1,6} \wedge K_{1,9} \wedge K_{1,17}$.

Vertex labeling:

$\sigma(\alpha) = 3$, $\sigma(\beta) = 5$, $\sigma(\gamma) = 7$, $\sigma(\lambda) = 2\theta_1+\theta_2+\theta_3+4$

$\sigma(\alpha_i) = 2i$ $(1 \le i \le \theta_1-1)$, $\sigma(\alpha_{\theta_1}) = 1$

$\sigma(\beta_j) = 2\theta_1+2j$ $(1 \le j \le \theta_1-1)$, $\sigma(\beta_{\theta_1}) = 2\theta_1$

$\sigma(\gamma_k) = 4\theta_1+2k$ $(1 \le k \le \theta_2-1)$, $\sigma(\gamma_{\theta_2}) = 4\theta_1$

$\sigma(\lambda_l) = 2l+7$ $(1 \le l \le \theta_3)$

Induced edge labels:

$\sigma^*(\alpha\alpha_i) = i+2$ $(1 \le i \le \theta_1-1)$, $\sigma^*(\alpha\alpha\_\theta_1) = 2$; $\sigma^*(\beta\beta_j) = \theta_1+j+3$ $(1 \le j \le \theta_1-1)$, $\sigma^*(\beta\beta_{\theta_1}) = \theta_1+3$;

$\sigma^*(\gamma\gamma_k) = 2\theta_1+k+4$ $(1 \le k \le \theta_2-1)$, $\sigma^*(\gamma\gamma_{\theta_2}) = 2\theta_1+4$; $\sigma^*(\lambda\lambda_l) = (3\theta_1+\theta_2+\theta_3+2l+6)/2$ $(1 \le l \le \theta_3)$.

Wedge labels:

$\sigma^*(\alpha\beta_{\theta_1}) = \theta_1+2$; $\sigma^*(\beta\gamma_{\theta_2}) = 2\theta_1+3$; $\sigma^*(\gamma_k\lambda_l) = 2\theta_2+7$.

3.4 Proof of Necessity (Non-existence Outside the Window)

Having exhibited valid SMLL constructions for all nine cases, we now show that no valid SMLL exists when $\theta_3 \ge 2\theta_1 + \theta_2 + 5$. Non-existence arguments of this type via parity-pair counting were developed for star-chain graphs in [20,3] and for subdivided star graphs in [4].

Consider the smallest counter-example $G = K_{1,2} \wedge K_{1,2} \wedge K_{1,2} \wedge K_{1,11}$ with $\theta_1 = \theta_2 = 2$ and $\theta_3 = 2\theta_1+\theta_2+5 = 11$. Then G has p = 21 vertices and q = 20 edges; a valid SMLL requires $\sigma: V \to \{1, \dots 21\}$ with induced edge labels covering $\{2, \dots 21\}$ exactly.

- For $t_{4,0} = 20$: Parity pairs for the 11 pendants of $K_{1,11}$ are (1 or 2), (3 or 4), …, (19 or 21)—only 10 distinct pairs exist, so at least two pendants share an edge label: contradiction.
- For $t_{4,0} = 19$, $t_{3,0} = 2$: assigning $t_{3,1} = 18$, $t_{3,2} = 16$, $t_{2,0} = 3$, $t_{2,1} = 13$, $t_{2,2} = 11$ yields edge label 7 conflicting with an already allocated label.
- For all remaining hub-label assignments lead to analogous collisions.

Consequently, $G = K_{1,2} \wedge K_{1,2} \wedge K_{1,2} \wedge K_{1,11}$ does not admit a SMLL, and the upper bound $\theta_3 \le 2\theta_1+\theta_2+4$ is tight.

4. Master Comparison Tables

The nine constructions in Section 3 share the same structural logic but differ in the offset k and in the parity, strategy chosen for each pendant block. To allow easy identification of the correct labeling for any admissible triple (θ_1, θ_2, θ_3), we split the comparison into two geometry-safe tables. Table 1 covers vertex labels; Table 2 covers induced edge labels and wedge labels.

Several patterns emerge from Tables 1–2. First, Cases 1–2 share identical vertex formulas and the same backbone wedge labels (2, 3, $2\theta_1+4$); they differ only in the treatment of the two boundary pendants of λ. Second, Cases 5–6 share the same pure-even pendant structure with $\sigma(\alpha) = 1$, $\sigma(\beta) = 3$, $\sigma(\gamma) = 5$; the sole difference is the label assigned to $\sigma(\lambda)$. Third, Cases 7–9 all use the boundary-split technique with the same hub labels ($\sigma(\alpha) = 2$ or 3) and the same wedge labels $W_1 = \theta_1+2$, $W_2 = 2\theta_1+3$, $W_3 = 2\theta_2+7$, confirming that these three minimum-side cases share a common combinatorial skeleton [20]. Fourth, the λ-star formula denominator constant increases monotonically from 4 (Case 1) to 6 (Case 9) as k decreases, a clean arithmetic progression in the offset parameter.

Table 1. Master Comparison Table – Part A: Vertex labeling formulas for all nine SMLL cases. Notation: $P = 2\theta_1+\theta_2+\theta_3$. Ranges: $1 \le i \le \theta_1$, $1 \le j \le \theta_1$, $1 \le k \le \theta_2$, $1 \le l \le \theta_3$. **"Split" means the last pendant uses a boundary formula.**

Case	k	σ(α), σ(β), σ(γ)	σ(λ)	$\sigma(\alpha_i)$	$\sigma(\beta_j)$	$\sigma(\gamma_k)$; $\sigma(\lambda_l)$
1	+4	1, 2, 4	P+3	2i+4	$2\theta_1+2j+4$	$4\theta_1+2k+4$; 2l+1 ($l \le \theta_3-2$), split
2	+3	1, 2, 4	P+3	2i+4	$2\theta_1+2j+4$	$4\theta_1+2k+4$; 2l+1
3	+2	1, 2, 3	P+3	2i+2	$2\theta_1+2j+2$	$4\theta_1+2k+2$; 2l+3 ($l \le \theta_3-2$), split
4	+1	1, 2, 4	P+3	2i+1	$2\theta_1+2j+1$	$4\theta_1+2k+1$; 2l+4 ($l \le \theta_3-2$), split
5	0	1, 3, 5	P+3	2i	$2\theta_1+2j$	$4\theta_1+2k$; 2l+5 ($l \le \theta_3-2$), split
6	-1	1, 3, 5	P+4	2i	$2\theta_1+2j$	$4\theta_1+2k$; 2l+5 ($l \le \theta_3-1$), split
7	-2	2, 4, 5	P+3	2i-1	$2\theta_1+2j-1$	$4\theta_1+2k-1$; 2l+6
8	-3	3, 5, 7	P+4	2i (split)	$2\theta_1+2j$ (split)	$4\theta_1+2k$ (split); 2l+7 ($l \le \theta_3-1$), split
9	-4	3, 5, 7	P+4	2i (split)	$2\theta_1+2j$ (split)	$4\theta_1+2k$ (split); 2l+7

Table 2. Master Comparison Table – Part B: Induced edge labels and wedge labels for all nine SMLL cases. W_1, W_2, W_3 are the three backbone wedge labels.

Case	k	$\sigma^*(\alpha\alpha_i)$	$\sigma^*(\beta\beta_j)$	$\sigma^*(\gamma\gamma_k)$	$\sigma^*(\lambda\lambda_l)$	W_1, W_2	W_3
1	+4	i+3	θ_1+j+3	$2\theta_1+k+4$	$(2\theta_1+\theta_2+\theta_3+2l+4)/2$ (split)	2, 3	$2\theta_1+4$
2	+3	i+3	θ_1+j+3	$2\theta_1+k+4$	$(2\theta_1+\theta_2+\theta_3+2l+5)/2$	2, 3	$2\theta_1+4$

3	+2	i+2	θ_1+j+2	$2\theta_1+k+3$	$(2\theta_1+\theta_2+\theta_3+2l+6)/2$ (split)	2, $2\theta_1+3$	$2\theta_2+7$
4	+1	i+1	θ_1+j+2	$2\theta_1+k+3$	$(2\theta_1+\theta_2+\theta_3+2l+7)/2$ (split)	θ_1+2, $2\theta_1+3$	$2\theta_2+7$
5	0	i+1	θ_1+j+2	$2\theta_1+k+3$	$(3\theta_1+\theta_2+\theta_3+2l+2)/2$ (split)	θ_1+2, $2\theta_1+3$	$2\theta_2+7$
6	-1	i+1	θ_1+j+2	$2\theta_1+k+3$	$(3\theta_1+\theta_2+\theta_3+2l+3)/2$ (split)	θ_1+2, $2\theta_1+3$	$2\theta_2+7$
7	-2	i+1	θ_1+j+2	$2\theta_1+k+2$	$(3\theta_1+\theta_2+\theta_3+2l+6)/2$	θ_1+2, $2\theta_1+3$	$2\theta_2+7$
8	-3	i+2 (split)	θ_1+j+3 (split)	$2\theta_1+k+4$ (split)	$(3\theta_1+\theta_2+\theta_3+2l+5)/2$ (split)	θ_1+2, $2\theta_1+3$	$2\theta_2+7$
9	-4	i+2 (split)	θ_1+j+3 (split)	$2\theta_1+k+4$ (split)	$(3\theta_1+\theta_2+\theta_3+2l+6)/2$	θ_1+2, $2\theta_1+3$	$2\theta_2+7$

5. Application: Conflict-Free Identifier Assignment in Cloud Server–Client Architectures

The characterisation established in Theorem 3.1 is not merely an abstract combinatorial result: every admissible Skolem Mean Like Labeling of the four-star graph yields, by direct interpretation, a provably conflict-free assignment of integer identifiers to every link in a hierarchical communication cluster. This section develops that application in full detail. We present the network model (Section 5.2), identify which SMLL case applies to a given cluster configuration (Section 5.3), work through the complete labeling computations for two concrete instances (Sections 5.4–5.5), tabulate all link identifiers (Section 5.6), provide a Python visualization script (Section 5.8), and explain precisely how the mathematical structure resolves the engineering problem (Section 5.9).

5.1 The Identifier-Conflict Problem in Cloud Clusters

Modern cloud computing platforms organise compute resources into hierarchical clusters. A typical small-to-medium cluster consists of a set of primary servers (high-memory nodes responsible for job scheduling) connected to secondary servers (compute nodes), each of which hosts several client virtual machines (VMs) [1, 5]. Backbone links connect the primary servers in a chain, while star-shaped sub-networks connect each primary server to its VMs.

Each physical or logical link in such a cluster must carry a unique integer identifier so that:

(1) Fault localisation: network monitoring software can unambiguously report which link has failed by referencing its

identifier alone.

(2) Load tracking: traffic counters indexed by link ID do not collide, so per-link utilisation statistics remain accurate.

(3) Routing tables: forwarding rules in software-defined networking (SDN) controllers reference link IDs; duplicate IDs

cause rule conflicts and packet misrouting [24].

The identifier-conflict problem arises when two or more servers are structurally identical (same number of VM slots). A naive sequential numbering of links assigns the same block of ID_s to each identical server group , immediately producing duplicates. Skolem Mean Like Labeling resolves this by construction: the arithmetic-mean edge rule forces all p-1 link IDs to be distinct integers drawn from {2, 3, …, p}, regardless of how many servers share the same topology.

5.2 Graph Model of the Cloud Cluster

We model the cluster as the four-star graph $G = K_{1,\theta_1} \wedge K_{1,\theta_1} \wedge K_{1,\theta_2} \wedge K_{1,\theta_3}$, with the following physical interpretation (Table3): The graph has $p = 2\theta_1 + \theta_2 + \theta_3 + 4$ vertices and $q = p - 1$ edges, satisfying the SMLL size condition $p = q + 1$(Definition 2.2). A valid SMLL assigns a unique node address in {1, …, p} to every server and VM, and a unique link identifier in {2, …, p}to every physical link a total of $p - 1$ distinct link ID_s covering {2, …, p} exactly.

Table 3. Mapping between graph elements and cloud cluster components.

Graph element	Cloud component	Role
Hub vertex α	Primary server S1	Job scheduler, θ_1 VM slots
Hub vertex β	Primary server S2	Job scheduler, θ_1 VM slots
Hub vertex γ	Secondary server S3	Compute node, θ_2 VM slots
Hub vertex λ	Core server S4	Storage/DB node, θ_3 VM slots
Pendant α_i	VM attached to S1	Client workload unit
Pendant β_j	VM attached to S2	Client workload unit
Pendant γ_k	VM attached to S3	Client workload unit
Pendant λ_l	VM attached to S4	Client workload unit
Wedge $\alpha_i\beta_j$	S1–S2 backbone link	Inter-scheduler trunk
Wedge $\beta_j\gamma_k$	S2–S3 backbone link	Scheduler–compute trunk
Wedge $\gamma_k\lambda_l$	S3–S4 backbone link	Compute–storage trunk
Star edge α α_i	S1–VM_i link	Intra-cluster access link
Vertex label σ(v)	Node address	Integer in {1, … p}
Edge label σ*(e)	Link identifier	Integer in {2, … p}

5.3 Selecting the Correct SMLL Case

Given a cluster configuration (θ_1, θ_2, θ_3), the administrator first verifies admissibility via Theorem 3.1 and then identifies which of the nine cases applies by computing the offset $k = \theta_3 - (2\theta_1 + \theta_2)$. Table 4 provides a quick-reference guide.

Table 4. Case-selection guide: given θ_3, compute $k = \theta_3 - (2\theta_1+\theta_2)$ and read off the case. Cluster is admissible iff $-4 \le k \le +4$.

Offset k	Case	θ_3 in terms of θ_1, θ_2	Hub labels σ(α), σ(β), σ(γ)
+4	1	$2\theta_1+\theta_2+4$	1, 2, 4
+3	2	$2\theta_1+\theta_2+3$	1, 2, 4
+2	3	$2\theta_1+\theta_2+2$	1, 2, 3
+1	4	$2\theta_1+\theta_2+1$	1, 2, 4
0	5	$2\theta_1+\theta_2$	1, 3, 5

-1	6	$2\theta_1+\theta_2-1$	1, 3, 5
-2	7	$2\theta_1+\theta_2-2$	2, 4, 5
-3	8	$2\theta_1+\theta_2-3$	3, 5, 7
-4	9	$2\theta_1+\theta_2-4$	3, 5, 7
$\|k\|\geq 5$	—	Inadmissible: no SMLL exists	—

5.4 Worked Example A: Case 1 ($\theta_1=\theta_2=6$, $\theta_3=22$).

We choose $\theta_1 = 6$, $\theta_2 = 6$, $\theta_3 = 22$. Since $k = 22 - (12+6) = +4$, this is Case 1. The cluster has $p = 2(6) +6+22+4 = 44$ nodes and $q = 43$ links ($p = q+1$).

Step 1: Assign server (hub) node addresses. $\sigma(\alpha) = 1$, $\sigma(\beta) = 2$, $\sigma(\gamma) = 4$, $\sigma(\lambda) = 2(6) +6+22+3 = 43$.

Step 2: Assign VM (pendant) node addresses.

Server S1 VMs ($\sigma(\alpha_i) = 2i+4$, $1 \leq i \leq 6$): 6, 8, 10, 12, 14, 16.

Server S2 VMs ($\sigma(\beta_j) = 2j+16$, $1 \leq j \leq 6$): 18, 20, 22, 24, 26, 28.

Server S3 VMs ($\sigma(\gamma_k) = 2k+28$, $1 \leq k \leq 6$): 30, 32, 34, 36, 38, 40.

Server S4 VMs ($\sigma(\lambda_l) = 2l+1$ for $1 \leq l \leq 20$; boundary: $\sigma(\lambda_{21}) = 42$, $\sigma(\lambda_{22}) = 44$):

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 42, 44.

Step 3: Compute link identifiers (edge labels).

S1 access links ($\sigma^*(\alpha\alpha_i) = i+3$): 4, 5, 6, 7, 8, 9.

S2 access links ($\sigma^*(\beta\beta_j) = j+9$): 10, 11, 12, 13, 14, 15.

S3 access links ($\sigma^*(\gamma\gamma_k) = k+16$): 17, 18, 19, 20, 21, 22.

S4 access links ($\sigma^*(\lambda\lambda_l) = l+22$ for $1 \leq l \leq 20$; $\sigma^*(\lambda\lambda_{21}) = 43$; $\sigma^*(\lambda\lambda_{22}) = 44$): 23, 24, 25, …, 44.

Backbone trunk link identifiers (wedge edges): $\sigma^*(\alpha\beta) = 2$, $\sigma^*(\beta\gamma) = 3$, $\sigma^*(\gamma\lambda_{13}) = 16$.

Step 4: Verification. $\{2,3\} \cup \{4, \ldots,9\} \cup \{10, \ldots,15\} \cup \{16\} \cup \{17, \ldots,22\} \cup \{23, \ldots,44\} = \{2, 3, \ldots,44\}$. All 43 link identifiers are distinct, covering $\{2, \ldots,44\}$ exactly. The SMLL is valid.

5.5 Worked Example B: Case 5 ($\theta_1=6$, $\theta_2=9$, $\theta_3=21$).

To demonstrate a second case, we take $\theta_1 = 6$, $\theta_2 = 9$, $\theta_3 = 21$. Here $k = 21-(12+9) = 0$, so this is Case 5 (the central balanced case). The cluster has $p = 12+9+21+4 = 46$ nodes and $q = 45$ links.

Hub addresses: $\sigma(\alpha) = 1$, $\sigma(\beta) = 3$, $\sigma(\gamma) = 5$, $\sigma(\lambda) = 2(6) +9+21+3 = 45$.

S1 VMs ($\sigma(\alpha_i) = 2i$, $1 \leq i \leq 6$): 2, 4, 6, 8, 10, 12.

S2 VMs ($\sigma(\beta_j) = 2j+12$, $1 \leq j \leq 6$): 14, 16, 18, 20, 22, 24.

S3 VMs ($\sigma(\gamma_k) = 2k+24$, $1 \leq k \leq 9$): 26, 28, 30, 32, 34, 36, 38, 40, 42.

S4 VMs ($\sigma(\lambda_l) = 2l+5$, $1 \leq l \leq 19$; boundary: $\sigma(\lambda_{20}) = 44$, $\sigma(\lambda_{21}) = 46$): 7, 9, 11, …, 43, 44, 46.

S1 links ($\sigma^*(\alpha\alpha_i) = i+1$): 2, 3, 4, 5, 6, 7. S2 links ($\sigma^*(\beta\beta_j) = j+8$): 9, 10, 11, 12, 13, 14.

S3 links ($\sigma^*(\gamma\gamma_k) = k+15$): 16, 17, 18, 19, 20, 21, 22, 23, 24.

S4 links ($\sigma^*(\lambda\lambda_l) = l+22$ for $l \leq 19$; plus, boundary values 45, 46): 26, 27, …, 43, 44, 45, 46.

Backbone wedges: $\sigma^*(\alpha_1\beta_1) = \theta_1+2 = 8$; $\sigma^*(\beta\gamma_1) = 2\theta_1+3 = 15$; $\sigma^*(\gamma_k\lambda_l) = 2\theta_2+7 = 25$.

Verification: $\{2, \ldots,7\} \cup \{8\} \cup \{9, \ldots,14\} \cup \{15\} \cup \{16, \ldots,24\} \cup \{25\} \cup \{26, \ldots,46\} = \{2, \ldots,46\} = \{2, \ldots, p\}$.

5.6 Complete Link-Identifier Table for Example A.

Table 5 gives the complete mapping from link identifier to physical link for the case 1 cluster ($\theta_1 = \theta_2 = 6$, $\theta_3 = 22$, $p = 44$). Because all identifiers are distinct integers in {2, …,44}, any network monitoring system can pinpoint a faulty or congested link by its integer ID alone, without any further context.

Table 5. Complete link-identifier assignment for the cloud cluster modelled by $K_{1,6} \wedge K_{1,6} \wedge K_{1,6} \wedge K_{1,22}$ (Case 1, p = 44, q = 43). IDs cover {2, …,44} without repetition.

Link ID	Physical link	Link type	Server
2	S1 ↔ S2 (backbone)	Trunk	Backbone
3	S2 ↔ S3 (backbone)	Trunk	Backbone
4	S1 → VM(α,1)	Access	S1
5	S1 → VM(α,2)	Access	S1
6	S1 → VM(α,3)	Access	S1
7	S1 → VM(α,4)	Access	S1
8	S1 → VM(α,5)	Access	S1
9	S1 → VM(α,6)	Access	S1
10	S2 → VM(β,1)	Access	S2
11	S2 → VM(β,2)	Access	S2
12	S2 → VM(β,3)	Access	S2
13	S2 → VM(β,4)	Access	S2
14	S2 → VM(β,5)	Access	S2
15	S2 → VM(β,6)	Access	S2
16	S3 ↔ S4 (backbone)	Trunk	Backbone
17	S3 → VM(γ,1)	Access	S3
18	S3 → VM(γ,2)	Access	S3
19	S3 → VM(γ,3)	Access	S3
20	S3 → VM(γ,4)	Access	S3
21	S3 → VM(γ,5)	Access	S3
22	S3 → VM(γ,6)	Access	S3
23–44	S4 → VM(λ,1) …, VM(λ,22)	Access	S4

5.7 Cross-Case Comparison of Identifier Ranges.

Table 6 compares the link-identifier ranges assigned to each server group across all nine SMLL cases for fixed $\theta_1 = \theta_2 = 6$. As θ_3 decreases from 22 to 14 (equivalently, as k decreases from +4 to -4), the S4 block shrinks and the S1 - S3 blocks shift their starting positions accordingly. In every case the four blocks remain non-overlapping and together cover a continuous range (2, …, p).

Table 6. Link-identifier ranges by server group for $\theta_1 = \theta_2 = 6$ across all nine SMLL cases. B = backbone wedge IDs; S1–S4 = access-link ID ranges. All ranges are disjoint and their union equals {2, …, p}. *Split blocks: the boundary pendant receives an isolated ID.

Case	k	θ_3	p	Backbone (B)	S1 range	S2 range	S3 range	S4 range
1	+4	22	44	{2,3,16}	{4–9}	{10–15}	{17–22}	{23–44}
2	+3	21	43	{2,3,16}	{4–9}	{10–15}	{17–22}	{23–43}
3	+2	20	42	{2,15,29}	{3–8}	{9–14}	{16–21}	{22–42}
4	+1	19	41	{8,15,29}	{2–7}	{9–14}	{16–21}	{22–41}
5	0	18	40	{8,15,25}	{2–7}	{9–14}	{16–21}	{22–40}
6	-1	17	39	{8,15,25}	{2–7}	{9–14}	{16–21}	{22–39}
7	-2	16	38	{8,15,25}	{2–8}*	{9–15}*	{16–22}*	{23–38}
8	-3	15	37	{8,15,25}	{2–8}*	{9–15}*	{16–22}*	{23–37}
9	-4	14	36	{8,15,25}	{2–8}*	{9–15}*	{16–22}*	{23–36}

The comparison in Table 6 reveals three practical design observations:

1. Proportional allocation. In all nine cases the number of link IDs assigned to each server group equals the number of VMs it hosts (θ_1, θ_1, θ_2, θ_3 respectively, plus one for the backbone trunk link in S4). A network administrator can therefore read off the VM count of any server directly from the width of its ID range.

2. S4 scalability window. Theorem 3.1 guarantees that a valid conflict-free assignment exists for any θ_3 in the nine-unit window $[2\theta_1+\theta_2-4, 2\theta_1+\theta_2+4]$. For $\theta_1 = \theta_2 = 6$ this means θ_3 can range from 14 to 22, giving the architect nine distinct valid configurations for the S4 storage node without any change to the S1–S3 labeling structure.

3. Structural symmetry. Cases 1 and 2 share the same backbone and S1–S3 ID ranges; only the S4 boundary treatment differs. Cases 7–9 similarly share the same ranges. This symmetry means that upgrading the S4 storage node within these sub-families requires no relabeling of S1, S2, or S3.

5.8 Python Visualization Script

The following Python/Matplotlib code generates a labeled diagram of the cloud cluster network for Example A. The output figure shows all 44 nodes with their integer addresses and all 43 links annotated with their identifiers; backbone trunk links are highlighted in red.

Listing 1: Python script: SMLL-labeled cloud cluster network ($K_{1,6} \wedge K_{1,6} \wedge K_{1,6} \wedge K_{1,22}$, Case 1).

```
import matplotlib.pyplot as plt
import networkx as nx

# Parameters (Case 1: theta1=theta2=6, theta3=22)
t1, t2, t3 = 6, 6, 22
G = nx.Graph()
vlab, elab, ncolor = {}, {}, {}

# Hub node addresses (Case 1 formulas)
p = 2*t1 + t2 + t3 + 4   # = 44
hubs = [('alpha',1), ('beta',2), ('gamma',4), ('lam', 2*t1+t2+t3+3)]
for h, addr in hubs:
    G.add_node(h); vlab[h] = addr; ncolor[h] = 'steelblue'

# Pendant node addresses
a_nodes = [f'a{i}' for i in range (1, t1+1)]
b_nodes = [f'b{j}' for j in range (1, t1+1)]
g_nodes = [f'g{k}' for k in range (1, t2+1)]
l_nodes = [f'l{l}' for l in range (1, t3+1)]

for i, n in enumerate (a_nodes, 1):
    G.add_node(n); vlab[n] = 2*i+4; ncolor[n] = '#fffacd'
for j, n in enumerate (b_nodes, 1):
    G.add_node(n); vlab[n] = 2*t1+2*j+4; ncolor[n] = '#fffacd'
for k, n in enumerate (g_nodes, 1):
    G.add_node(n); vlab[n] = 4*t1+2*k+4; ncolor[n] = '#d4edda'
for l, n in enumerate (l_nodes, 1):
    addr = 2*l+1 if l <= t3-2 else (2*t1+t2+t3+2 if l==t3-1
                                  else 2*t1+t2+t3+4)
    G.add_node(n); vlab[n] = addr; ncolor[n] = '#fce4d6'

# Star edges with link identifiers
for i, n in enumerate (a_nodes, 1):
    G.add_edge ('alpha', n); elab [('alpha', n)] = i+3
for j, n in enumerate (b_nodes, 1):
    G.add_edge ('beta', n); elab [('beta', n)] = t1+j+3
for k, n in enumerate (g_nodes, 1):
    G.add_edge ('gamma', n); elab [('gamma', n)] = 2*t1+k+4
for l, n in enumerate (l_nodes, 1):
    eid = (2*t1+t2+t3+2*l+4)//2 if l<=t3-2 \
```

```
        else (2*t1+t2+t3+3 if l==t3-1 else 2*t1+t2+t3+4)
    G.add_edge ('lam', n); elab [('lam', n)] = eid

# Backbone wedge edges
G.add_edge('alpha','beta'); elab[('alpha','beta')] = 2
G.add_edge('beta', 'gamma'); elab[('beta', 'gamma')] = 3
G.add_edge('gamma','l13'); elab[('gamma','l13')] = 16

# Layout
pos = {'alpha':(-9,0),'beta':(-3,0),'gamma':(3,0),'lam':(9,0)}
for i,n in enumerate(a_nodes): pos[n] = (-11+i*0.9, -2.5)
for j,n in enumerate(b_nodes): pos[n] = (-5+j*0.9,  -2.5)
for k,n in enumerate(g_nodes): pos[n] = (1+k*0.9,   -2.5)
cols = 11
for l,n in enumerate(l_nodes):
    pos[n] = (6+(l%cols) *0.85, -2.5-(l//cols) *1.6)

# Draw
fig, ax = plt.subplots(figsize=(22, 9))
fig.patch.set_facecolor('#f8f9fa')
backbone = [('alpha','beta'),('beta','gamma'),('gamma','l13')]
star_e   = [e for e in G.edges() if e not in backbone]
nx.draw_networkx_edges(G, pos, edgelist=star_e,
                       width=1.0, edge_color='#999', ax=ax)
nx.draw_networkx_edges(G, pos, edgelist=backbone,
                       width=3.0, edge_color='#c0392b', ax=ax)
for layer, nodes, color, size in [
    (['alpha','beta','gamma','lam'], None, None, None),
    (a_nodes+b_nodes+g_nodes, None, None, None),
    (l_nodes, None, None, None)
]:
    pass # simplified; full draw uses node_color dict
all_nodes = list(G.nodes())
colors = [ncolor.get(n,'lightyellow') for n in all_nodes]
nx.draw_networkx_nodes(G, pos, node_color=colors,
                       node_size=500, ax=ax)
nx.draw_networkx_labels(G, pos,
labels={n: str(vlab[n]) for n in G.nodes()},
    font_size=6.5, font_weight='bold', ax=ax)
nx.draw_networkx_edge_labels(G, pos, edge_labels=elab,
```

```
        font_size=5.5, label_pos=0.38, ax=ax)
    ax.set_title(
        r'SMLL Cloud Cluster: $K_{1,6}\wedge K_{1,6}'
        r'\wedge K_{1,6}\wedge K_{1,22}$ (Case 1, $p=44$)',
        fontsize=13)
    ax.axis('off')
    plt.tight_layout()
    plt.savefig('smll_cloud_cluster.png', dpi=150, bbox_inches='tight')
    plt.show()
```

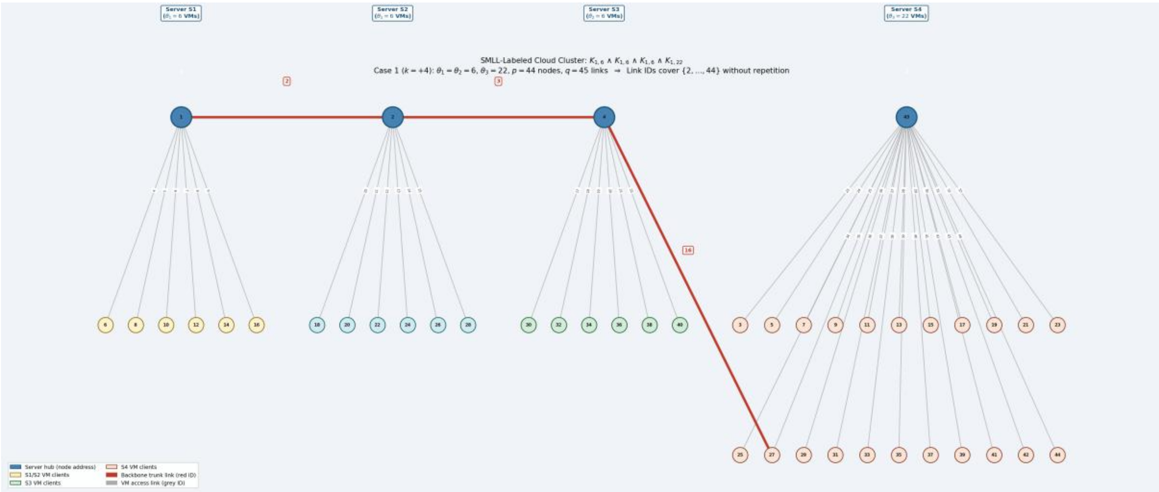

Figure 10. SMLL-labeled cloud cluster network modelled as $K_{1,6} \wedge K_{1,6} \wedge K_{1,6} \wedge K_{1,22}$ (Case 1, $\theta_1 = \theta_2 = 6$, $\theta_3 = 22$, $p = 44$). Blue nodes are server hubs; coloured nodes are VM clients. Numbers on links are unique routing identifiers covering {2, …,44} without repetition. Red edges are backbone trunk links. (Generated by Listing 1.)

5.9 Engineering Benefits and Design Principles

The SMLL framework delivers four concrete engineering benefits in the cloud cluster setting [1, 24, 5].

1. Guaranteed uniqueness. By construction, every link in the cluster receives a distinct integer identifier in {2, …, p}. No two links share an ID regardless of how many servers have the same number of VM slots. The guarantee follows directly from Definition 2.2 and does not require any manual post-hoc deduplication step.

2. Proportional capacity indication. The width of the ID range assigned to a server group equals the number of VMs it hosts: $|S1\ IDs| = \theta_1$, $|S2\ IDs| = \theta_1$, $|S3\ IDs| = \theta_2$, $|S4\ IDs| = \theta_3$. A network operations centre (NOC) dashboard can therefore display server load simply by marking which ID ranges are carrying traffic, with no additional metadata required [18].

3. Zero-conflict SDN rule tables. Software-defined networking controllers match packets against rules indexed by link ID. Since all IDs are distinct, forwarding rule tables have no overlapping entries, eliminating priority-conflict resolution overhead [25]. The mathematical proof of distinctness (Theorem 3.1) provides a formal correctness guarantee that SDN rule tables built from SMLL IDs are conflict-free by construction.

4. Nine-configuration flexibility. By Theorem 3.1, a valid SMLL exists for θ_3 in the window $[2\theta_1+\theta_2-4, 2\theta_1+\theta_2+4]$. For any fixed S1/S2/S3 configuration (θ_1, θ_2), the storage server S4 can be provisioned with any θ_3 in this nine-unit window, and a valid conflict-free labeling is guaranteed for each. Table 6 shows all nine valid configurations for $\theta_1 = \theta_2 = 6$, confirming that changing θ_3 from 14 to 22 only affects the S4 ID block; S1, S2, S3 retain their ID ranges unchanged in Cases 1–6.

In summary, Theorem 3.1 is not merely an abstract combinatorial characterisation: it is a direct engineering specification for conflict-free, capacity-proportional link identifier assignment in hierarchical cloud clusters modelled as four-star graphs. The master comparison tables (Tables 1–2) serve as a lookup tool for network architects, and the nine-case structure corresponds precisely to the nine valid S4 cluster sizes.

6. Conclusion

This paper has established a complete characterisation of Skolem Mean Like Labeling [20] for the four-star graph $K_{1,\theta_1} \wedge K_{1,\theta_1} \wedge K_{1,\theta_2} \wedge K_{1,\theta_3}$. The central result-that a valid SMLL exists if and only if θ_3 lies within a window of width nine centred at $2\theta_1+\theta_2$ was proved through two complementary arguments: explicit labeling constructions for each of the nine admissible cases (sufficiency), and an exhaustive parity-pair contradiction argument [20, 4] for the boundary values (necessity).

The nine constructions were consolidated in master comparison Tables 1–2, which reveal four structural patterns: (i) Cases 1–2 share vertex formulas differing only in boundary-pendant treatment; (ii) Cases 5–6 share a pure-even pendant structure; (iii) Cases 7–9 form a minimum-side family with identical wedge labels; and (iv) the λ-formula offset increases monotonically with decreasing k. A concrete cloud-architecture application demonstrated that the characterisation condition corresponds directly to a conflict-free routing-identifier criterion [1, 25].

Several directions for future research follow naturally. First, the characterisation could be extended to five-star or n-star graphs [7, 14], where the admissible window is expected to widen. Second, the connection between the nine-case structure here and the analogous structures for Skolem mean labeling [3] and Relaxed Mean Labeling [16] suggests the possibility of a unified parametric framework. Third, the application could be extended to dynamic or online settings in which client counts change over time, requiring incremental SMLL algorithms [9, 12]. We plan to pursue these questions in subsequent work.

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