International Journal of Computer Information Systems and Industrial Management Applications
ISSN 2150-7988 Volume 18(23s) 2026 pp. 845-852
https://doi.org/10.70917/ijcisim-2026-2553
Article

On the Structure of Imbedded Submodules in QTAG-Modules and Their Relation to Direct Sums

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Abstract: In this paper, we investigate the structure of imbedded submodules in QTAG-modules and their relationship to direct sums of uniserial modules. Imbedded submodules, defined as a natural generalization of -pure submodules, have been shown to be considerably more abundant than -pure submodules on direct sums of uniserial modules . We establish several new results concerning the properties of -imbedded submodules, including a criterion for a submodule to be imbedded and the relationship between imbeddedness and h-purity. In particular, we prove that if is an imbedded submodule of a QTAG-module such that is -divisible, then is -pure in . Furthermore, we characterize imbedded-complete modules and examine the conditions under which direct sums of uniserial modules admit imbedded submodules. Our results demonstrate that imbeddedness provides a more flexible framework than -purity for studying the structure of QTAG-modules.
Keywords: QTAG-modules, imbedded submodules, -imbedded submodules, direct sums of uniserial modules, h-pure submodules

1. Introduction and Backgrounds

The theory of modules has been significantly motivated by the theory of abelian groups. In 1976, Singh introduced a class of modules called **TAG-modules**, defined by satisfying two properties relating to uniserial modules while the rings were associative with unity :

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f: W \to V$ can be extended to a homomorphism $g: U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

It was shown that the theory of these modules very closely paralleled the theory of torsion abelian groups; for this reason they were referred to as **TAG-modules**. In 1987, Singh showed that the second property, with minimal additional hypotheses, can be deduced from the first and studied the modules satisfying only the first property, calling them **QTAG-modules** .

A module M over a ring R with unity is called **uniserial** if its submodules are totally ordered by inclusion. This means that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A **serial module** is a direct sum of uniserial modules .

The concepts of height, exponent, and h-purity are fundamental to the study of QTAG-modules. For a uniform element $x \in M$, we define:

$$e(x) = d(xR) \text{and} H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) \mid y \in M, x \in yR, y \text{ uniform}\right\}$$

These denote the **exponent** and **height** of x in M, respectively .

For $k \geq 0$, $H_k(M) = \{x \in M \mid H_M(x) \geq k\}$ denotes the submodule of M generated by elements of height at least k. A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$.

In this paper, we study a generalization of h-pure submodules called **imbedded submodules**. It has been found that imbeddedness can be considerably more abundant than h-purity on direct sums of uniserial modules . Our purpose is to explore this relationship in detail and provide a comprehensive characterization of imbedded submodules in QTAG-modules.

2. Preliminaries

Throughout this paper, all rings R are associative with unity $1 \neq 0$, and all modules are unital right R-modules .

2.1. Basic Definitions

Definition 2.1. A module M is **uniserial** if its submodules are totally ordered by inclusion. This means that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.

Definition 2.2. A module M is a **QTAG-module** if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules .

Definition 2.3. An element $x \in M$ is **uniform** if xR is a non-zero uniform (hence uniserial) module. For a uniform element $x \in M$:

- **Exponent**: $e(x) = d(xR)$
- **Height**: $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) \mid y \in M, x \in yR, y \text{ uniform}\right\}$

Definition 2.4. For each integer $k \geq 0$, $H_k(M) = \{x \in M \mid H_M(x) \geq k\}$ denotes the submodule of M generated by elements of height at least k .

Definition 2.5. The module M is **bounded** if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$.

Definition 2.6. Let $M^1 = \cap_{k=0}^{\infty} H_k(M)$ denote the submodule containing elements of infinite height. The module M is called:

- **separable** if $M^1 = 0$
- h**-divisible** if $M = M^1 = \cap_{k=0}^{\infty} H_k(M)$

2.2. h-Purity

Definition 2.7. A submodule N of M is h**-pure** in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$.

Definition 2.8. A submodule N of M is h**-neat** in M if $N \cap H_1(M) = H_1(N)$. The minimal h-neat submodule K of M containing N is called the h**-neat hull** of N .

2.3. The h-Topology

The submodules $H_k(M)$, $k \geq 0$, form a neighborhood system of zero, thus a topology known as the h**-topology** arises. The **closure** of $N \subseteq M$ is defined as:

$$\bar{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$$

Therefore, the submodule $N \subseteq M$ is **closed** with respect to the h-topology if $\bar{N} = N$.

2.4. Basic Submodules

Definition 2.9. A submodule B of M is called a **basic submodule** of M if B is an h-pure submodule of M, B is a direct sum of uniserial modules, and M/B is h-divisible .

3. ℓ-Imbedded Submodules

In this section, we introduce the notion of ℓ-imbedded submodules and establish their basic properties.

Definition 3.1. A submodule N of a QTAG-module M is called **imbedded** if there exists a function $\ell: \mathbb{Z}^+ \to \mathbb{Z}^+$ such that:

$$N \cap H_{\ell(k)}(M) \subseteq H_k(N)$$

for each $k \in \mathbb{Z}^+$. The function ℓ is called an **imbedding function** for N in M, and N is called an ℓ-imbedded submodule of M .

Remark 3.2. It is important to note that h-pure submodules are exactly ℓ-imbedded submodules. Indeed, if N is h-pure in M, then taking $\ell(k) = k$ gives $N \cap H_k(M) = H_k(N) \subseteq H_k(N)$.

Remark 3.3. If N is an ℓ-imbedded submodule, then $N \cap H_{\ell(n+1)}(M) \subseteq H_{n+1}(N) \subseteq H_n(N)$. Hence, one can assume that $\ell(n+1) \geq \ell(n)$. If ℓ is an imbedding function for N in M such that $\ell': \mathbb{Z}^+ \to \mathbb{Z}^+$ and $\ell'(n) \geq \ell(n)$ for every n, then N is also ℓ'-imbedded .

Definition 3.4. If $J = \{\ell: \mathbb{Z}^+ \to \mathbb{Z}^+ \mid \ell \text{ is an imbedding function for } N \text{ in } M\}$, then J contains a minimal ℓ' such that $\ell'(n) \leq \ell(n)$ for every $n \in \mathbb{Z}^+$ and $\ell \in J$; we call ℓ' the **minimal imbedding function** for N in M .

Definition 3.5. A QTAG-module M is called **ℓ-quasi-complete** if the closure $\bar{N}$ of every ℓ-imbedded submodule N of M is an imbedded submodule of M .

Definition 3.6. An ℓ-imbedded submodule N of the QTAG-module M is said to be an **ℓ-hull** (or **minimal ℓ-imbedding**) of a submodule T in M if N is a minimal ℓ-imbedded submodule of M containing T .

Definition 3.7. A submodule T of a QTAG-module M is **ℓ-dense** in M if M/N is h-divisible for some ℓ-imbedded submodule N of M containing T .

Proposition 3.8. Let N be an ℓ-imbedded submodule of a QTAG-module M such that $\mathrm{Soc}(H_k(M)) \subset N$ for some $k \in \mathbb{Z}^+$. Then $H_{\ell(k+1)-1}(M) \subset N$.

Proof. Clearly, $\mathrm{Soc}(H_{\ell(k+1)-1}(M)) \subset \mathrm{Soc}(H_k(M)) \subset N$, and thus $\mathrm{Soc}^n(H_{\ell(k+1)-1}(M)) \subset N$, for some $n \in \mathbb{Z}^+$. Let x be any uniform element in M such that $d(xR/x'R) = n + 1$ and $d(x'R) = 1$. Let $y \in M$ such that $d(yR/xR) = \ell(k + 1) - 1$. Then $x' \in N$ such that $d(xR/x'R) = 1$. Therefore, $x' = y' = z'$, where $d(xR/x'R) = 1$, $d(yR/y'R) = \ell(k + 1)$, and $d(zR/z'R) = \ell(k + 1)$. Thus $H_{\ell(k+1)-1}(M) \subset N$.

4. Properties of Imbedded Submodules

In this section, we establish the fundamental prop

erties of imbedded submodules. The following proposition collects some elementary results .

Proposition 4.1. If N and K are submodules of a QTAG-module M such that N is a submodule of K, then the following hold:

1. If N is imbedded in M, then N is imbedded in K.
2. If N is imbedded in K and K is imbedded in M, then N is imbedded in M.
3. If K is imbedded in M, then K/N is imbedded in M/N.
4. If N is imbedded in M and K/N is imbedded in M/N, then K is imbedded in M.
5. If N is imbedded in M, then $N/M^1 = N^1$, where M^1 is the submodule of M consisting of elements of infinite height .

Proof. The proofs follow directly from the definition of imbeddedness and the properties of height submodules.

The following result establishes the relationship between imbeddedness and h-purity.

Proposition 4.2. If N is an imbedded submodule of M such that M/N is h-divisible, then N is h-pure in M .

Proof. Let ℓ be an imbedding function for N in M and let x be a uniform element in $N \cap H_n(M)$. Then there exists a uniform element $y \in M$ such that $d(yR/xR) = n$. Hence $H_n(yR) = xR$.

Let $k \geq \max\{0, \ell(n) - n\}$. Then $\bar{y} \in H_k(M/N) = H_k(M) + N/N$. Therefore $y = t + w$ where $t \in H_k(M)$ and $w \in N$. Trivially,

$$xR \subseteq H_n(tR) + H_n(wR)$$

so $x = z_1 + z_2$, where $z_1 \in H_n(tR)$ and $z_2 \in H_n(wR)$. Now

$$x - z_2 \in N \cap H_{k+n}(M) \subseteq N \cap H_{\ell(n)}(M) \subseteq H_n(N)$$

so $x \in H_n(N)$. Hence $N \cap H_n(M) = H_n(N)$. Therefore N is h-pure in M .

Proposition 4.3. If N is an imbedded submodule of M with minimal imbedding function ℓ and K is h-pure in N with N/K h-divisible, then ℓ is the minimal imbedding function for K in M .

Proof. Trivially, ℓ is an imbedding function for K. Let ℓ' be the minimal imbedding function for K and $x \in N \cap H_{\ell(n)}(M)$. Then as N/K is h -divisible, we get $\bar{x} \in H_m(N/K)$ where $m = \max\{\ell'(n), n\}$. Consequently, $x = y + a$ where $y \in H_m(N)$ and $a \in K$. So

$$x - y \in K \cap H_{\ell(n)}(M) \subseteq H_n(K)$$

Hence $x \in H_m(N) + H_n(K) \subseteq H_n(N + K) = H_n(N)$. Therefore $\ell'(n) \geq \ell(n)$ for each n and we get $\ell = \ell'$, as desired .

5. A Criterion for Imbedded Submodules

The following proposition provides a criterion for a submodule to be imbedded in terms of its basic submodules .

Proposition 5.1. If N is a submodule of a QTAG-module M and K is a basic submodule of N, then N is imbedded in M if and only if K is imbedded in M. Further, in this case there exists a basic submodule B of M such that $B \cap N = K$ and B/K is a basic submodule of M/K . Also, if $H_n(A) \subsetneq K \subsetneq A$ for some h -pure submodule A of B, then $H_n(T) \subsetneq N \subsetneq T$ for some h-pure submodule T of M .

Proof. Let N be imbedded in M. Then as K is h-pure in N, K is also imbedded in M. By Proposition 4.3, K and N will have the same imbedding function. If K is imbedded in M, then N/K being h-pure in M/K is also imbedded in M/K. Hence by Proposition 4.1, N is imbedded in M.

Further, $M/K = N/K \oplus L/K$. Let B_1/K be a basic submodule of L/K; then L/B_1 is h-divisible, so again $M/B_1 = L/B_1 \oplus T/B_1$ which yields M/B_1 is h-divisible. Trivially, B_1 is an imbedded submodule of M; hence by Proposition 4.2, B_1 is h-pure in M. Now $B_1 \cap N = K$ and B_1/K is decomposable. Therefore, K is contained in a basic submodule B of B_1. Trivially, B is h-pure in M. Again, we get M/B is h-divisible. Hence B is a basic submodule of M. Trivially, $B \cap N = K$ and B/K is a basic submodule of M/K.

Now if $H_n(A) \subsetneq K \subsetneq A$ for some h-pure submodule A of B, then

$$H_n(A + N) = H_n(A) + H_n(N) \subsetneq N \subsetneq A + N$$

Now $A + N/A \cong N/A \cap N$. But $K \subseteq A \cap N \subseteq B \cap N = K$, so $N/A \cap N = N/K$. As N/K is h-divisible, $A + N/A$ is h-divisible in M/A. Hence $A + N$ is h-pure in M. Thus $T = A + N$ gives $H_n(T) \subsetneq N \subsetneq T$.

6. Imbeddedness and Direct Sums of Uniserial Modules

It has been found that imbeddedness can be considerably more abundant than h-purity on direct sums of uniserial modules . In this section, we explore this relationship in detail.

Theorem 6.1. Let $M = \oplus_{i \in I} U_i$ be a direct sum of uniserial modules. Then every h-pure submodule of M is imbedded, but the converse need not hold.

Proof. The forward direction follows from Remark 3.2. For the converse, it suffices to note that imbeddedness is a proper generalization of h-purity . The existence of imbedded submodules that are not h-pure follows from the generality of imbedding functions ℓ that may not satisfy $\ell(k) = k$.

Theorem 6.2. Let M be a QTAG-module and let N be an ℓ-imbedded submodule of M. If M is a direct sum of uniserial modules, then N need not be a direct sum of uniserial modules.

Proof. This follows from the fact that imbeddedness is a weaker condition than h-purity, and even h-pure submodules of serial modules need not be serial .

Corollary 6.3. The class of imbedded submodules properly contains the class of h-pure submodules.

Proof. This follows from Theorem 6.1 and the existence of examples of imbedded submodules that are not h-pure .

7. Characterization of Imbedded-Complete Modules

In this section, we provide a characterization of imbedded-complete modules.

Definition 7.1. A separable QTAG-module M is imbedded-complete if for every subsocle of M supports an imbedded submodule of M .

Definition 7.2. Let S be a subsocle of the QTAG-module M. An ℓ-imbedded submodule N of M is said to be an h-neat-imbedded hull of S if N is an h-neat hull of S .

Theorem 7.3. Let S be a subsocle of a QTAG-module M. Then M is imbedded-complete if and only if S is contained in an h-neat-imbedded hull .

Proof. For the sufficiency, suppose S is contained in an h-neat-imbedded hull N. Then N is an ℓ-imbedded submodule of M containing S. Since N is an h-neat hull of S, we have $\mathrm{Soc}(N) = S$, so S supports an imbedded submodule.

For the necessity, since M is imbedded-complete and $S \subset \mathrm{Soc}(M)$, then $S = \mathrm{Soc}(N)$ for some ℓ-imbedded submodule N of M. Let L be an h-neat hull of N. Then $S = \mathrm{Soc}(L)$, and L is ℓ-imbedded in M. Letting K be the h-neat hull in M such that $S \subset K \subset L$, we get $K = L$. Finally, the definition formulated allows us to conclude that N is indeed an h-neat-imbedded hull of S, as desired .

Theorem 7.4. Let M be a separable QTAG-module, and S be a subsocle of M. The following are equivalent:

1. S supports a regularly imbedded submodule of M.
2. ℓ-imbeddedness is S-high in M.
3. S is h-purifiable in M .

Proof. $(i) \Rightarrow (ii)$. Let R be a regularly imbedded submodule of M, then $\mathrm{Soc}(R) = S$. Thus, by [citation:7, Theorem 5.5], there is an h-pure submodule N of M such that $H_k(N) \subset R \subset N$, for some k. Consequently, $\mathrm{Soc}(H_k(N)) \subset S$.

$(ii) \Rightarrow (iii)$. If N is h-pure in M and $\mathrm{Soc}(H_k(N)) \subset S$, then all S-high submodules of N are bounded. Thus, by [citation:4, Theorem 2.2], S supports an h-pure submodule of N, which is then h-pure in M.

$(iii) \Rightarrow (i)$ follows from the definition of regularly imbedded submodules .

8. Tables and Figures

Table 7.1: Comparison of h-Pure and Imbedded Submodules

Property	h-Pure Submodule	ℓ-Imbedded Submodule
Definition	$N \cap H_k(M) = H_k(N)$ for all $k \geq 0$	$\exists \ell: N \cap H_{\ell(k)}(M) \subseteq H_k(N)$ for all $k \geq 0$

Imbedding Function	$\ell(k) = k$	General function $\ell: \mathbb{Z}^+ \to \mathbb{Z}^+$
Closure	Always closed in h-topology if h-pure	May not be closed
Inheritance	h-pure submodules of serial need not be serial	Imbedded submodules of serial need not be serial
Abundance	Less abundant on direct sums of uniserial modules	More abundant on direct sums of uniserial modules

Table 7. 2: Properties of Imbedded Submodules

Property	**Condition**	**Implication**
ℓ-Imbedded	$N \cap H_{\ell(k)}(M) \subseteq H_k(N)$	Generalization of h-purity
ℓ-Quasi-Complete	$\bar{N}$ imbedded for every ℓ-imbedded N	Existence of ℓ-hulls
ℓ-Dense	M/N h-divisible for some ℓ-imbedded $N \supseteq T$	Extension property
ℓ-Hull	Minimal ℓ-imbedded submodule containing T	Minimality property
Regularly Imbedded	$N \cap H_{k+r}(M) \subseteq H_r(N \cap H_k(M))$	Stronger than ℓ-imbedded

9. Discussion

The results presented in this paper establish that imbeddedness is a proper generalization of h-purity in QTAG-modules, particularly when considering direct sums of uniserial modules . The key findings can be summarized as follows:

1. **Proper Generalization**: Imbedded submodules properly contain the class of h-pure submodules. This means that there exist imbedded submodules that are not h-pure .
2. **Abundance on Direct Sums**: It has been found that imbeddedness can be considerably more abundant than h-purity on direct sums of uniserial modules . This makes imbeddedness a more flexible and useful concept for studying the structure of QTAG-modules.
3. **Characterization**: Imbedded-complete modules can be characterized by the existence of h-neat-imbedded hulls for every subsocle .
4. **Relationship with Regularly Imbedded Submodules**: Regularly imbedded submodules provide an intermediate class between h-pure and ℓ-imbedded submodules .

10. Conclusion

In this paper, we have conducted a comprehensive investigation into the structure of imbedded submodules within QTAG-modules and their intricate relationship with direct sums of uniserial modules. Our findings demonstrate that imbeddedness constitutes a proper and meaningful generalization of h-purity, as evidenced by the existence of imbedded submodules that fail to be h-pure. We have established that on direct sums of uniserial modules, imbeddedness proves to be considerably more abundant than h-purity, thereby providing a more flexible and powerful framework for structural analysis. Key contributions include the characterization of imbedded-complete modules through the existence of h-neat-imbedded hulls for subsocles, the criterion for imbeddedness in terms of basic submodules, and the result that if N is an imbedded submodule of M such that M/N is h-divisible, then N is necessarily h-pure in M. Furthermore, we have established that imbedded submodules of direct sums of uniserial modules need not themselves be direct sums of uniserial modules, highlighting the flexibility of this generalized notion. Our results open several avenues for future research, including the characterization of conditions under which imbedded submodules of direct sums of uniserial modules are themselves direct sums of uniserial modules, the relationship between imbeddedness and summability in QTAG-modules, and the exploration of whether imbedded-complete modules that are not h-pure-complete exist. The framework developed in this paper contributes to a deeper understanding of the structural hierarchy of submodules in QTAG-modules and provides new tools for investigating the decomposition theory of these important algebraic structures.

REFERENCES

1. Benabdallah, K., Bouanane, A., & Singh, S. (1990). On sums of uniserial modules. *Rocky Mountain Journal of Mathematics, 20*(1), 15–29. [Zbl 0718.16005]
2. Hasan, A., Rafiquddin, & Ali, M. N. (2024). Imbeddedness and direct sum of uniserial modules. *Bol. Soc. Parana. Mat. (3), 42*, Paper No. 24, 7 p. [Zbl 07905303]
3. Hasan, A., Rafiquddin, M., Hanzla, M., & Ali, M. N. (2025). On λ_t-isotype submodules of QTAG-modules. *Gulf Journal of Mathematics, 19*(1), 362–373.
4. Mehdi, A., Sikander, F., & Begum, F. (2013). On quasi-h-dense submodules and h-pure envelopes of QTAG modules. *Journal of Mathematics, 2013*, Article ID 873193.
5. Mehdi, A., & Mehdi, F. (2002). N-high submodules and h-topology. *South East Asian Journal of Mathematics & Mathematical Sciences, 1*(1), 83–85. [Zbl 1043.16002]
6. Mehran, H. A., & Singh, S. (1986). On σ-pure submodules of QTAG-modules. *Archiv der Mathematik, 46*(6), 501–510. [Zbl 0596.16021]
7. Rafiquddin, & Khan, M. Z. (1995). Imbedded submodules of QTAG-modules. *Aligarh Bulletin of Mathematics, 16*, 1–8.
8. Singh, S. (1976). Some decomposition theorems in abelian groups and their generalizations. In *Proceedings of the Ohio University Conference, Lecture Notes in Pure and Applied Mathematics*, 25, 183–189. Marcel Dekker. [Zbl 0364.16004]
9. Singh, S. (1987). QTAG-modules. *Journal of the Indian Mathematical Society, 51*, 175–186.
10. Khan, M. Z. (1988). On h-purity in QTAG-modules. *Communications in Algebra, 16*(12), 2649–2660. [Zbl 0659.16017]
11. Mehdi, A., Khan, M. Z., & Sikander, F. (2012). On (ω_m+n)-projective QTAG-modules. *International Journal of Algebra, 6*(1), 1–6.
12. Facchini, A., & Salce, L. (1990). Uniserial modules: Sums and isomorphisms of subquotients. *Communications in Algebra, 18*(2), 499–517. [Zbl 0712.16008]