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Input-Output Stability Of Neural Network With Hysteresis

G.Padmavathi[1],J.Neelima[1], P.V.Siva Kumar[2],G.V.Deepika[1]

[1]C.R.Rao Advanced Institute of Mathematics Statistics and Computer Science,
University of Hyderabad campus,Hyderabad, India.
padmavathi@crraoaimscs.res.in.

[2]VNRVignana Jyothi Institute of Engineering & Technology,
Bachupally, Nizampet (SO),Hyderabad – 500090, India.
sivakumar_pv@vnrvjiet.in

Abstract— **In this paper analysis of hysteresis neural network towards stability are proposed. In the present research existence, asymptotic stability and Input-output stability of equations as a model for hysteretic neurons are discussed. These neural networks can also be employed for image extraction in a noise interfering channels. We establish sufficient conditions for various stability analysis of this class of neural networks. The result improves the earlier publications due to the Input output analysis of the network with neutral delays.**

Keywords- Hysteresis Neural Networks, Asymptotic stability, Input output stability.

I. INTRODUCTION

Hysteresis can be observed in many engineering systems such as control systems, electronic circuits and also observed in animals such as frogs [15] and crayfish [9]: In a piezoelectric actuator; hysteresis means that for a certain input; there is no unique output and the output depends on the input history [7]. Mathematical models describing the dynamical interactions of hysteresis neural network have been discussed in ([2],[4-8]). In this paper we consider the class of continuous – time hysteresis neural network as a model described by the following form of neutral delay differential equations

$$\dot{z}_i = -a_i z_i(t) + \sum_{j=1}^{n} b_{ij} f_j \begin{pmatrix} z_j(t) + m_j z_j(t-\tau) \\ +q_j \dot{z}_j(t-\sigma) + \alpha_j \end{pmatrix}, \qquad \textbf{(1)}$$

$t > 0. Here\ \dot{z}_i = \dfrac{dz_i(t)}{dt},\ \ i = 1,2,...n,\ j = 1,2,...,n.$

The initial functions associated with the system (1) are given by $z_i(s) = \pi_i(s)$ for $s \in [-\gamma, 0]$ for each $i = 1,2,..,n$, here $\gamma = \max\{\sigma, \tau\}$. $\pi_i \in C'([-\gamma,0], R)$,for each $i = 1,2,...n.$

$$Let \quad x_i = z_i(t) + m_i z_i(t-\tau) + q_i \dot{z}_i(t-\sigma) + \alpha_i, \quad for\ \ t > 0. \qquad \textbf{(2)}$$

Then equation (1) can be written as

$$\dot{z}_i = -a_i z_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)), \qquad \textbf{(3)}$$

Differentiating w.r.t't' (2) and using system (3),

$$\begin{aligned}\frac{d}{dt}[x_i(t)] &= -a_i z_i(t) + \sum_{j=1}^{n} b_{ij} f_j(z_j(t)) \\ &+ l_i\left(-a_i z_i(t-\tau) + \sum_{j=1}^{n} b_{ij} f_j(z_j(t-\tau))\right) \\ &+ q_i \frac{d}{dt}\left(-a_i z_i(t-\sigma) + \sum_{j=1}^{n} b_{ij} f_j(z_j(t-\sigma))\right),\end{aligned}$$

then system (1) can be written in the following form which is mathematically convenient to work with

$$\begin{aligned}&\frac{d}{dt}\left[x_i(t) - q_i \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\sigma))\right] \\ &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) \\ &+ l_i \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau)) + I_i,\ \ I_i = a_i \alpha_i\ .\end{aligned} \qquad \textbf{(4)}$$

$For\ any\ s \in [-\gamma, 0], i = 1,2,..,n$,
$x_i(s) = \pi_i(s) + l_i \pi_i(s-\tau) + q_i \dot{\pi}_i(s-\sigma) + \alpha_i = \phi_i(s).$

From $\pi_i \in C'([-\gamma,0], R)$ we have $\phi_i \in C'([-\gamma,0], R)$.

Therefore, $x_i(t) = \phi_i(s), \text{for } s \in [-\gamma, 0]$ are the initial conditions associated with the network (2).

The system (1) may be viewed as first order differential equations of neutral type with varying inputs. From network (1) we can observe that function f_i, depends not only on the output of a system, but also history of the rate of change of its output. Input-output representation and state variable representation are two different behaviors of looking at the same [20]. The two types of representations are used as each of them give a different kind of approach into how the system works .There exists a very close relationship between the types of stability results .Hence one can find adopting these two approaches. The latter approach is aimed at the determination of output bounds given the characteristics of the feedback system and its input. Both the input and the output bounds are defined in some normed spaces. Thus, the issue of input-output stability is referred to as an Lp stability analysis. Lp stability theory has been extensively studied in the literature ([12], [16],[17],[20],[23]). On the other hand, the techniques of functional analysis, pioneered by Sandberg [16-17] and Zames [23] have developed equally rapidly and generated a large number of results concerning the input-output properties of nonlinear feedback systems. The Lp stability of linear feedback systems with a single time-varying sector-bounded element is studied in [13]. The subject of feedback systems stability has been extensively dwelt upon in the literature [20].

In the present investigation we establish results on dealing with the circumstances under which conditions x of (4) is L^p - stable.

Definition 1. The solution $x = 0$ is L^p - stable for (4), if it is stable and for $(t_0, u_0) \in D_M$ where

$$D_M = \{(t, x) / t \geq 0, |x| < M\}, 0 < M \leq +\infty.$$

Let $F(t, t_0, u_0)$ be any solution of (4) for which $F(t, t_0, u_0) = u_0$, for all $t_0 \geq 0$,there exists a $\delta_0 = \delta_0(t_0) > 0$ such that if $|u_0| < \delta_0$ then

$$\int_{t_0}^{\infty} |F(t, t_0, u_0)|^p \, dt < \infty. \, .$$

II. EXISTENCE AND UNIQUENESS

It is easy to see that the equilibrium of the system (4) is a solution of the following system of equations. For $i = 1, 2, ..., n,$

$$a_i x_i^* = \sum_{j=1}^{n} b_{ij} (1 + l_i) \, f_j \left(x_{ij}^* \right) + I_i \quad . \tag{5}$$

Throughout this discussion, we assume that the functions f_i satisfy the following conditions:

For $i = 1, 2, ..., n,$ there exist positive quantities L_i such that

$$\left| f_i(u(t)) - f_i(v(t)) \right| \leq L_i |u - v|, \tag{6}$$

for $u, v \in R$ and $t \in [0, \infty)$..

Now our first result is concerned with the existence of a unique equilibrium x_i^* for the system (4), for $i = 1, 2, ..., n,$

Theorem 1. Assume that condition (6) satisfied. In addition assume that the decay rates a_i, the synaptic weights b_{ij}, and the parameters L_i, satisfy the following inequality

$$L_i \sum_{j=1}^{n} |b_{ji}| \, \frac{(|l_j| + 1)}{a_j} < 1. \tag{7}$$

Then under these conditions there exist a unique equilibrium point for the system (4).

Proof. If $x = x^*$ is an equilibrium point with $x^* = (x_1^*, x_2^*, ..., x_n^*)^T$, x^* satisfies the following

$$x_i^* = \sum_{j=1}^{n} b_{ij} \frac{(1 + l_i)}{a_i} f_j(x_j^*) + \alpha_i \ , \quad \text{for each i= } 1, 2, \ldots \text{n.} \tag{8}$$

Define a mapping

$$H = \{H_{ij}, i = 1, 2, ...n; \; j = 1, 2, ...n.\}$$

Where

$$H_{ij} = x_i \to \left(-x_i + \sum_{j=1}^{n} b_{ij} \frac{(1 + l_i)}{a_i} f_j(x_j) + \alpha_i \right), \quad \text{for each } x \in R^n. \tag{9}$$

In [14], a locally invertible C_0 map $H : R^n \to R^n$ is a homeomorphism of R^n onto itself if it is proper that is $H^{-1}(K)$ is compact for any compact set K in R^n. So if we verify that H is proper we can have that H is a bijective mapping and hence H(x) = 0 has a unique solution. Therefore (4) has another set of unique equilibrium solution for each $x \in R^n$. First we prove that if (4) have equilibrium then it is unique. Suppose that there are two equilibrium points x^* and $\tilde{x}^*$.

Then we can obtain

$$\left|x_i^* - \tilde{x}_i^*\right| \le \sum_{j=1}^{n} \left|b_{ij}\right| \frac{1+\left|l_i\right|}{a_i} \left|f_j\left(x_j^*\right) - f_j\left(\tilde{x}_j^*\right)\right|, \quad i = 1, 2, \ldots n. \qquad \textbf{(10)}$$

Thus we have

$$\sum_{i=1}^{n} \left|x_i^* - \tilde{x}_i^*\right| \le \sum_{i=1}^{n} L_i \left(\sum_{j=1}^{n} \left|b_{ji}\right| \frac{1+\left|l_j\right|}{a_j} \right) \left|x_i^* - \tilde{x}_i^*\right|, \; i = 1, 2, \ldots n.$$

This implies we have

$$\sum_{i=1}^{n} \left(1 - L_i \left(\sum_{j=1}^{n} \left|b_{ji}\right| \frac{1+\left|l_j\right|}{a_j} \right) \right) \left|x_i^* - \tilde{x}_i^*\right| \le 0. \qquad (11)$$

If we assume

$$L_i \left(\sum_{j=1}^{n} \left|b_{ji}\right| \frac{1+\left|l_j\right|}{a_j} \right) < 1 \; . \qquad \textbf{(12)}$$

We have $\left|x_i^* - \tilde{x}_i^*\right| = 0.$ Therefore $x_i^* = \tilde{x}_i^*$. Hence if (4) have equilibrium, it has unique equilibrium. To show the existence of the equilibrium of (4) it is enough to show that H is a homeomorphism of R^n onto itself. From the uniqueness of an equilibrium proof we have that if $x^* \neq \tilde{x}^*$ then we have $H(x^*) \neq H(\tilde{x}^*)$, hence H is one-to-one. Therefore H is locally invertible C_0 mapping. To prove proper it sufficient to prove that that H is

$$\left\|H(x)\right\| \to \infty \quad for \quad \left\|x\right\| \to \infty.$$

Consider

$$\begin{aligned} \bar{H}_{ij}(x_i) &= H_{ij}(x_i) - H_{ij}(0) \\ &= \left(-x_i + \sum_{i=1}^{n} b_{ij} \frac{(1+l_i)}{a_i} \left[f_j\left(x_j\right) - f_j\left(0\right) \right] \right). \end{aligned}$$

Thus we have

$$\begin{aligned} \sum_{i=1}^{n} \bar{H}_{ij}(x_i) &= \sum_{i=1}^{n} \left| -x_i + \sum_{j=1}^{n} b_{ij} \frac{(1+l_i)}{a_i} \left[f_j\left(x_j\right) - f_j\left(0\right) \right] \right| \\ &\ge \sum_{i=1}^{n} \left[\left| x_i \right| - \sum_{j=1}^{n} \left|b_{ij}\right| \frac{1+\left|l_i\right|}{a_i} \left| f_j\left(x_j\right) - f_j\left(0\right) \right| \right] \\ &\ge \sum_{i=1}^{n} \left[1 - L_i \sum_{j=1}^{n} \left|b_{ji}\right| \frac{1+\left|l_j\right|}{a_j} \right] \left| x_i \right|. \end{aligned}$$

Let $\varepsilon = \min\limits_{i=1,2,..,n} \left\{ 1 - L_i \sum_{j=1}^{n} \left|b_{ji}\right| \frac{1+\left|l_j\right|}{a_j} \right\}.$

From (10) we have $\varepsilon > 0$ and hence $\left\|\bar{H}(x)\right\| \ge \varepsilon \left\|x\right\|$ which implies we have $\lim\limits_{\|x\| \to \infty} \left\|\bar{H}(x)\right\| = \infty.$ So we have $\lim\limits_{\|x\| \to \infty} \left\|H(x)\right\| = \infty.$ Hence $H(x) = 0$ has unique solution and (4) has a unique equilibrium point.

III. STABILITY ANALYSIS

Global asymptotic stability of an equilibrium means that the recall is perfect in the sense no hints or guesses are needed. Now we recall that the equilibrium x^* associated with input I is globally asymptotically stable independent of delays, if every solution x of (4) corresponding to an arbitrary choice of initial functions (6), satisfies $\lim\limits_{n \to \infty} x(t) = x^*$. .

Lemma 1 [21]. Let $f\left(t\right) \in c\left[a, \infty\right]$. If $\int_a^\infty f^2(t) < \infty$ and $f'\left(t\right)$ is bounded, then $f\left(t\right) \to 0$ as $t \to \infty$.

Theorem 2. Assume that the following inequalities are satisfied

for each i=1,2,..,n. $L_i \left(\sum_{j=1}^{n} \left|b_{ji}\right| \left|q_i\right| \right) < 1$.

Further there exist an $\eta > 0$ such that

$$\begin{aligned} &a_i \left(2 - \left|q_i\right| \eta^{-1} \sum_{j=1}^{n} \left|b_{ij}\right| L_j \right) \\ &> \left(\sum_{j=1}^{n} \left|b_{ij}\right| L_j \eta^{-1} \left(1 + \left|l_i\right|\right) + L_i \eta \sum_{j=1}^{n} \left|b_{ji}\right| \left(1 + \left|l_j\right| + c_j \left|q_j\right|\right) \right. \\ &\left. + L_i^2 \left(\eta + \eta^{-1}\right) \left(\sum_{j=1}^{n} \left|b_{ji}\right| \left|q_j\right| \right) \left(\sum_{j=1}^{n} \left|b_{ji}\right| \left(1 + \left|l_j\right|\right) \right) \right), \end{aligned} \qquad \textbf{(13)}$$

Then the unique equilibrium solution x^* of (4) is globally asymptotically stable.

Proof. From (4) and (5) we have

$$\begin{aligned} &\frac{d}{dt} \left[(x_i\left(t\right) - x_i^*) - q_i \sum_{j=1}^{n} b_{ij} (f_j\left(x_j\left(t - \sigma\right)\right) - f_j\left(x_j^*\right)) \right] \\ &= -a_i (x_i\left(t\right) - x_i^*) + \sum_{j=1}^{n} b_{ij} (f_j\left(x_j\left(t\right)\right) - f_j\left(x_j^*\right)) \\ &+ l_i \sum_{j=1}^{n} b_{ij} (f_j\left(x_j\left(t - \tau\right)\right) - f_j\left(x_j^*\right)) \, , \end{aligned} \qquad \textbf{(14)}$$

Consider

$$E(x)(t)$$
$$=\sum_{i=1}^{n}\left\{\left[(x_i(t)-x_i^*)-q_i\sum_{j=1}^{n}b_{ij}(f_j(x_j(t-\sigma))-f_j(x_j^*))\right]^2\right.$$
$$+\left[L_i\eta\sum_{j=1}^{n}|b_{ji}||q_j|a_j+L_i^2\eta\left(\sum_{j=1}^{n}|b_{ji}||q_j|\right)\left(\sum_{j=1}^{n}|b_{ji}|\right)\right.$$
$$\left.+L_i^2\eta\left(\sum_{j=1}^{n}|b_{ji}||q_j|\right)\left(\sum_{j=1}^{n}|b_{ji}||l_j|\right)\right]\int_{t-\sigma}^{t}|x_j(s)-x_j^*|^2\,ds$$
$$+\left[\begin{array}{l}L_i\eta\left(\sum_{j=1}^{n}|b_{ji}||l_j|\right)\\+L_i^2\eta^{-1}\left(\sum_{j=1}^{n}|b_{ji}||q_j|\right)\left(\sum_{j=1}^{n}|b_{ji}||l_j|\right)\end{array}\right]\int_{t-\tau}^{s}|x_j(s)-x_j^*|^2\,ds. \quad \textbf{(15)}$$

Differentiating with respect to 't' and Using (14), (6) and using the inequality $2ab\le\eta a^2+\eta^{-1}b^2$,for any $\eta>0$ for all real a, b we have

$$\frac{dE(x)(t)}{dt}\le\sum_{i=1}^{n}\left\{-2a_i(|x_i(t)-x_i^*|^2\right.$$
$$+\sum_{j=1}^{n}|b_{ij}|L_j\left(\eta|x_j(t)-x_j^*|^2+\eta^{-1}|x_i(t)-x_i^*|^2\right)$$
$$+|l_i|\sum_{j=1}^{n}|b_{ij}|L_j\left(\eta|x_j(t-\tau)-x_j^*|^2+\eta^{-1}|x_i(t)-x_i^*|^2\right)$$
$$+|q_i|\sum_{j=1}^{n}|b_{ij}|L_j\left(\eta|x_j(t-\sigma)-x_j^*|^2+\eta^{-1}|x_i(t)-x_i^*|^2\right)a_i$$
$$+L_i^2\left(\sum_{j=1}^{n}|b_{ji}||q_j|\right)\left(\sum_{j=1}^{n}|b_{ji}|\right)\left(\begin{array}{l}\eta|x_i(t-\sigma)-x_i^*|^2\\+\eta^{-1}|x_i(t)-x_i^*|^2\end{array}\right)$$
$$+L_i^2\left(\sum_{j=1}^{n}|b_{ji}||q_j|\right)\left(\sum_{j=1}^{n}|b_{ji}||l_j|\right)\left(\begin{array}{l}\eta|x_i(t-\sigma))-x_i^*|^2\\+\eta^{-1}|x_i(t-\tau)-x_i^*|^2\end{array}\right)$$
$$+\left[L_i\eta\sum_{j=1}^{n}|b_{ji}||q_j|a_j+L_i^2\eta\left(\sum_{j=1}^{n}|b_{ji}||q_j|\right)\left(\sum_{j=1}^{n}|b_{ji}|\right)\right.$$
$$\left.+L_i^2\eta\left(\sum_{j=1}^{n}|b_{ji}||q_j|\right)\left(\sum_{j=1}^{n}|b_{ji}||l_j|\right)\right]$$
$$\times\left[|x_i(t)-x_i^*|^2-|x_i(t-\sigma)-x_i^*|^2\right]$$
$$+\left[L_i\eta\left(\sum_{j=1}^{n}|b_{ji}||l_j|\right)+L_i^2\eta^{-1}\left(\sum_{j=1}^{n}|b_{ji}||q_j|\right)\left(\sum_{j=1}^{n}|b_{ji}||l_j|\right)\right]$$
$$\left.\times\left[|x_i(t)-x_i^*|^2-|x_i(t-\tau)-x_i^*|^2\right]\right\}.$$
$$\textbf{(16)}$$

Rearranging the terms we have

$$\frac{dE(x)(t)}{dt} \le \sum_{i=1}^{n}\Big\{-2a_i(\left|x_i(t)-x_i^*\right|^2$$
$$+\eta\sum_{j=1}^{n}\left|b_{ji}\right|L_j\left|x_i(t)-x_i^*\right|^2+\sum_{j=1}^{n}\left|b_{ij}\right|L_j\eta^{-1}\left|x_i(t)-x_i^*\right|^2$$
$$+\eta L_i\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\left|x_i(t-\tau)-x_i^*\right|^2$$
$$+\left|l_i\right|\eta^{-1}\sum_{j=1}^{n}\left|b_{ij}\right|L_j\left|x_i(t)-x_i^*\right|^2+\left|q_i\right|\eta^{-1}\sum_{j=1}^{n}\left|b_{ij}\right|L_j a_i\left|x_i(t)-x_i^*\right|^2$$
$$+L_i\eta\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|a_j\left|x_i(t-\sigma)-x_i^*\right|^2$$
$$+L_i^2\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|\right)\left(\sum_{j=1}^{n}\left|b_{ji}\right|\right)\eta\left|x_i(t-\sigma)-x_i^*\right|^2$$
$$+L_i^2\eta^{-1}\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|\right)\left(\sum_{j=1}^{n}\left|b_{ji}\right|\right)\left|x_i(t)-x_i^*\right|^2$$
$$+L_i^2\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|\right)\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\eta\left|x_i(t-\sigma)-x_i^*\right|^2$$
$$+L_i^2\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|\right)\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\eta^{-1}\left|x_i(t-\tau)-x_i^*\right|^2$$
$$+\left[L_i\eta\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|a_j+L_i^2\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|\right)\left(\sum_{j=1}^{n}\left|b_{ji}\right|\right)\eta\right.$$
$$\left.+L_i^2\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|\right)\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\eta\right]\left[\left|x_i(t)-x_i^*\right|^2-\left|x_i(t-\sigma)-x_i^*\right|^2\right]$$
$$+\left[L_i\eta\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)+L_i^2\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|\right)\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\eta^{-1}\right]$$
$$\times\left[\left|x_i(t)-x_i^*\right|^2-\left|x_i(t-\tau)-x_i^*\right|^2\right]\Big\}. \quad \textbf{(17)}$$

This intern leads to

$$\frac{dE(t,x)}{dt} \le -\sum_{i=1}^{n}\mu_i\left|x_i(t)-x_i^*\right|^2. \quad \textbf{(18)}$$

Where

$$\mu_i = a_i\left(2-\left|q_i\right|\eta^{-1}\sum_{j=1}^{n}\left|b_{ij}\right|L_j\right)$$
$$-\left[\eta^{-1}\sum_{j=1}^{n}\left|b_{ij}\right|L_j\left(1+\left|l_i\right|\right)+L_i\sum_{j=1}^{n}\left|b_{ji}\right|\eta\left(1+\left|l_j\right|+a_j\left|q_j\right|\right)\right.$$
$$\left.+L_i^2\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|\right)\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left(\eta+\eta^{-1}\right)\left(1+\left|l_j\right|\right)\right)\right],$$

Thus we have

$$\frac{dE(x)(t)}{dt} \le 0, \quad t>0.$$

Since $\frac{dE(x)(t)}{dt} \le -\sum_{i=1}^{n}\mu_i\left|x_i(t)-x_i^*\right|^2$.

Thus we have

$$E(x)(t)+\sum_{i=1}^{n}\mu_i\int_0^t\left|x_i(s)-x_i^*\right|^2 ds \le E(x)(0).$$

Since $E(x)(t)\ge 0$, we have

$$\sum_{i=1}^{n}\mu_i\int_0^t\left|x_i(s)-x_i^*\right|^2 ds \le E(x)(0).$$

$$\sum_{i=1}^{n}\int_0^t\left|x_i(s)-x_i^*\right|^2 ds \le \infty. \quad \textbf{(19)}$$

Now we prove that $\left\|x-x^*\right\|'$, are bounded.

Since $\sum_{i=1}^{n}\mu_i\int_0^t\left|x_i(s)-x_i^*\right|^2 ds \ge 0.$

From (19) we have $E(x)(t)\le E(x)(0)$.

Thus for each $i=1,2,\ldots n.$

$$\left|(x_i(t)-x_i^*)-q_i\sum_{j=1}^{n}b_{ij}(f_j(x_j(t-\sigma))-f_j(x_j^*))\right| \le \sqrt{E(x)(0)}.$$

So

$$\left|(x_i(t)-x_i^*)\right| \le \sqrt{E(x)(0)} + \left|q_i\sum_{j=1}^{n}b_{ij}(f_j(x_j(t-\sigma))-f_j(x_j^*))\right|.$$

Using (6) we have

$$\left|x_i(t)-x_i^*\right| \le \sqrt{E(x)(0)} + L_j\left|q_i\right|\sum_{j=1}^{n}\left|b_{ij}\right|\left|x_j(t-\sigma)-x_j^*\right|.$$

This implies we have

$$\sup_{s\in[-\gamma,0]}\left|(x_i(s)-x_i^*)\right| \le \sqrt{E(x)(0)} + \sup_{s\in[-\gamma,0]}\left|x_i(s)-x_i^*\right|L_i\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|q_j\right|\right).$$

Thus we have

$$\sup_{s\in[-\gamma,0]}\left|(x_i(s)-x_i^*)\right|\le \frac{\sqrt{E(x)(0)}}{1-L_i\left(\sum_{j=1}^{n}|b_{ji}||q_j|\right)}. \quad \textbf{(20)}$$

Therefore $\left|x_i(s)-x_i^*\right|$ is uniform bounded. Thus $\left\|x(s)-x^*\right\|'$, is also bounded.

Without loss of generality, we can assume that $I_i=0$ in equation (4), and $f_i(0)=0,\ i=1,2,....n$ to make it computationally tractable and to provide in depth analysis.

The following lemmas can be used in the derivation of L^p - stability.

Lemma 2: Let $1\le p<\infty$ .Then for a, b, η nonnegative we have

(i) $(a+b)^p\le 2^p(a^p+b^p)$.

(ii) $(a+\eta b)^p\ge a^p+p\eta ba^{p-1}$

$hence\ \ p\eta ba^{p-1}\le 2^p(a^p+\eta^p b^p).$

Lemma 3: If $0<p<\infty$, put $\gamma_p=\max\{1,2^{p-1}\}$ then for arbitrary complex numbers α,β ,

$$|\alpha-\beta|^p\le\gamma_p(|\alpha|^p+|\alpha|^p).$$

Definition 2: The norm of an element $y=(y_1,y_2,...,y_n)$ of Euclidean n-space E^n is given by $|y|=\sum_{i=1}^{n}|y_i|$.

Lemma 4 [1]. Let V be a Liapunov function on E such that $V_{(E)}(t,u)\le -c|u|^p$ on

$$D_M=\left\{(t,\xi)/t\ge 0,|\xi|<M\right\},0<M\le+\infty.$$

for some $c>0,\ p>0$. Then $u=0$ is L^P - stable for (A) [Where $u^1=f(t,u)$ (A)].

Lemma 4[1]: If there exists $V(t,u)$, a Liapunov function then $u=0$ is stable for (A) .

Theorem 3: Assume that the conditions for $p>1$

$$\begin{aligned}
a_i\gamma_p\left(p-2^p\gamma_p^{p-1}\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}L_j^{p-1}\right)\ge 2^p\sum_{j=1}^{n}|b_{ij}|L_j\left(1+|l_i|\right)\\
+2^pL_i\gamma_p^p\sum_{j=1}^{n}|b_{ji}|+2^pL_i^p\gamma_p^p\left(\sum_{j=1}^{n}|b_{ji}|\right)\left(\sum_{j=1}^{n}|q_jb_{ji}|^{p-1}\right)\\
+2^p\left[\begin{array}{l}L_i^p\left(\sum_{j=1}^{n}|b_{ji}|(1+|l_j|)\right)\left(\sum_{j=1}^{n}|q_jb_{ji}|^{p-1}\right)\\ +L_i^{p-1}\left(\sum_{j=1}^{n}|q_jb_{ji}|^{p-1}a_j\right)\end{array}\right]\\
+2^pL_i\left(\sum_{j=1}^{n}|b_{ji}||l_j|\right)\left[1+L_i^{p-1}\left(\sum_{j=1}^{n}|q_jb_{ji}|^{p-1}\right)\right].
\end{aligned}\quad \textbf{(21)}$$

Then the $x=0$ is L^P - stable for the system (4).

Proof.

Let $E(t,x)=\sum_{i=1}^{n}E_i(t,x)$, where

$$E_i(t,x)=\left|x_i(t)-q_i\sum_{j=1}^{n}b_{ij}f_j\left(x_j(t-\sigma)\right)\right|^p,t>0.\ , \quad \textbf{(22)}$$

Clearly $E_i(t,x)$ is positive definite, $E_i(t,0)=0$, $t\ge 0$ and $E_i(t,x)$ is continuous on D_M .Now we verify that $E_i(t,x)$ is locally Lipschitzian on D_M^* .For any $(t,x),(t,\tilde{x})\in D_M^*=D_M-\{(t,0)/t\ge 0\}$,

Consider

$$\begin{aligned}
&E_i(t,x)-E_i(t,\tilde{x})\\
&\le p\left[\begin{array}{l}\left|x_i(t)-q_i\sum_{j=1}^{n}b_{ij}\ f_j\left(x_j(t-\sigma)\right)\right|^{p-1}\\ +\left|\tilde{x}_i(t)-q_i\sum_{j=1}^{n}b_{ij}\ f_j\left(\tilde{x}_j(t-\sigma)\right)\right|^{p-1}\end{array}\right] \quad \text{U}\\
&\times\left[\left(x_i(t)-\tilde{x}_i(t)\right)-q_i\sum_{j=1}^{n}b_{ij}\left(f_j\left(x_j(t-\sigma)\right)-f_j\left(\tilde{x}_j(t-\sigma)\right)\right)\right].
\end{aligned}$$

sing the inequality for any r_1,r_2 ,

$\left|r_1^P-r_2^P\right|\le P\left(r_1^{P-1}+r_2^{P-1}\right)|r_1-r_2|$ we have

$$E_i(t,x)-E_i(t,\tilde{x}) \le p\begin{bmatrix} \left|x_i(t)-q_i\sum_{j=1}^{n}b_{ij}\, f_j\left(x_j(t-\sigma)\right)\right|^{p-1} \\ +\left|\tilde{x}_i(t)-q_i\sum_{j=1}^{n}b_{ij}\, f_j\left(\tilde{x}_j(t-\sigma)\right)\right|^{p-1}\end{bmatrix} \times\left[\left(x_i(t)-\tilde{x}_i(t)\right)-q_i\sum_{j=1}^{n}b_{ij}\left(f_j\left(x_j(t-\sigma)\right)-f_j\left(\tilde{x}_j(t-\sigma)\right)\right)\right].$$

Using lemma 3 and inequality (6) we have

$$E_i(t,x)-E_i(t,\tilde{x}) \le p\eta_p\begin{bmatrix} |x_i(t)|^{p-1}+\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}L_j^{p-1}|x_j(t-\sigma)|^{p-1} \\ +|\tilde{x}_i(t)|^{p-1}+\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}L_j^{p-1}|x_j(t-\sigma)|^{p-1}\end{bmatrix} \times\left[|x_i(t)-\tilde{x}_i(t)|+|q_i|\sum_{j=1}^{n}|b_{ij}|L_j|x_j(t-\sigma)-\tilde{x}_j(t-\sigma)|\right]$$

where $\eta_p=\max\{1,2^{p-2}\}$. From the definition of D_M^* ,we have $|x|<M,|\tilde{x}|<M$,

$$E_i(t,x)-E_i(t,\tilde{x}) \le p\eta_p\begin{bmatrix} M^{p-1}+\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}L_j^{p-1}M^{p-1} \\ +M^{p-1}+\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}L_j^{p-1}M^{p-1}\end{bmatrix} \times\left[|x_i(t)-\tilde{x}_i(t)|+|q_i|\sum_{j=1}^{n}|b_{ij}|L_j|x_j(t-\sigma)-\tilde{x}_j(t-\sigma)|\right]$$

$$\le 2p\eta_pM^{p-1}\kappa_i\left(1+L_i\sum_{j=1}^{n}|b_{ji}q_i|\right)|x-\tilde{x}|.$$

where $\kappa_i=1+\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}L_j^{p-1}$.

Thus we have $E_i(t,x)$ is locally Lipschitzian on D_M^* .

Differentiate equation (22) w.r.t 't' and using equation (4) and lemma 3 we have

$$\frac{dE_i(t,x)}{dt}=p\left|x_i(t)-q_i\sum_{j=1}^{n}b_{ij}\, f_j\left(x_j(t-\sigma)\right)\right|^{p-1} \times\frac{d}{dt}\left|x_i(t)-q_i\sum_{j=1}^{n}b_{ij}f_j\left(x_j(t-\sigma)\right)\right|$$

$$\le p\gamma_p[|x_i(t)|^{p-1}+\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}|f_j(x_j(t-\sigma))|^{p-1}]$$

$$[-a|x_i(t)|+\sum_{j=1}^{n}|b_{ij}||f_j(x_j(t))|+|l_i|\sum_{j=1}^{n}|b_{ij}||f_j(x_j(t-\tau))|],$$

where $\gamma_p=\max\{1,2^{p-2}\}$. .

Simplifying we have

$$\frac{dE_i(t,x)}{dt}\le p\gamma_p[|x_i(t)|^{p-1}\begin{bmatrix}-a_i|x_i(t)|+\sum_{j=1}^{n}|b_{ij}||f_j(x_j(t))| \\ +|l_i|\sum_{j=1}^{n}|b_{ij}||f_j(x_j(t-\tau))|\end{bmatrix}$$

$$+\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}|f_j(x_j(t-\sigma))|^{p-1}\begin{bmatrix}-a_i|x_i(t)|+\sum_{j=1}^{n}|b_{ij}||f_j(x_j(t))| \\ +|l_i|\sum_{j=1}^{n}|b_{ij}||f_j(x_j(t-\tau))|\end{bmatrix}.$$

Thus we have

$$\frac{dE_i(t,x)}{dt}\le -a_i|x_i(t)|\,p\gamma_p|x_i(t)|^{p-1}$$
$$+p\gamma_p|x_i(t)|^{p-1}\sum_{j=1}^{n}|b_{ij}||f_j(x_j(t))|$$
$$+p\gamma_p|x_i(t)|^{p-1}|l_i|\sum_{j=1}^{n}|b_{ij}||f_j(x_j(t-\tau))|$$
$$-a_i|x_i(t)|\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}|f_j(x_j(t-\sigma))|^{p-1}$$
$$+\left(\sum_{j=1}^{n}|b_{ij}||f_j(x_j(t))|\right)\left(\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}|f_j(x_j(t-\sigma))|^{p-1}\right)$$
$$+\left(|l_i|\sum_{j=1}^{n}|b_{ij}||f_j(x_j(t-\tau))|\right)\left(\sum_{j=1}^{n}|q_ib_{ij}|^{p-1}|f_j(x_j(t-\sigma))|^{p-1}\right).$$

From (6) and rearranging the terms we have

$$
\begin{aligned}
\frac{dE_i(t,x)}{dt} &\le -a_i p\gamma_p \left|x_i(t)\right|^p + p\gamma_p \sum_{j=1}^{n}\left|b_{ij}\right| L_j \left|x_j(t)\right|^{p-1}\left|x_i(t)\right| \\
&+ p\gamma_p \left|l_i\right| \sum_{j=1}^{n}\left|b_{ij}\right| L_j \left|x_j(t)\right|^{p-1}\left|x_i(t-\tau)\right| \\
&+ p\gamma_p a_i \sum_{j=1}^{n}\left|q_i b_{ij}\right|^{p-1} L_j^{p-1}\left(\left|x_i(t)\right|\left|x_j(t-\sigma)\right|^{p-1}\right) \\
&+ p\gamma_p L_i^p \left(\sum_{j=1}^{n}\left|b_{ji}\right|\right)\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right)\left|x_i(t)\right|\left|x_i(t-\sigma)\right|^{p-1} \\
&+ p\gamma_p L_i^p \left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right)\left|x_i(t-\tau)\right|\left|x_i(t-\sigma)\right|^{p-1}.
\end{aligned}
$$

Using the inequality $p\gamma_p b a^{p-1} \le 2^p(a^p + \gamma_p^p b^p)$ for any non- negative a,b we have

$$
\begin{aligned}
\frac{dE_i(t,x)}{dt} &\le -a_i p\gamma_p \left|x_i(t)\right|^p + 2^p \sum_{j=1}^{n}\left|b_{ij}\right| L_j \left(\left|x_j(t)\right|^p + \gamma_p^p\left|x_i(t)\right|^p\right) \\
&+ 2^p \left|l_i\right| \sum_{j=1}^{n}\left|b_{ij}\right| L_j \left(\left|x_i(t)\right|^p + \gamma_p^p\left|x_j(t-\tau)\right|^p\right) \\
&+ 2^p a_i \sum_{j=1}^{n}\left|q_i b_{ij}\right|^{p-1} L_j^{p-1}\left(\left|x_j(t-\sigma)\right|^p + \gamma_p^p\left|x_i(t)\right|^p\right) \\
&+ 2^p L_i^p \left(\sum_{j=1}^{n}\left|b_{ji}\right|\right)\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right)\left(\left|x_j(t-\sigma)\right|^p + \gamma_p^p\left|x_i(t)\right|^p\right) \\
&+ 2^p L_i^p \left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right)\left(\left|x_j(t-\sigma)\right|^p + \gamma_p^p\left|x_i(t-\tau)\right|^p\right).
\end{aligned}
\qquad \textbf{(23)}
$$

Simplifying, we have

$$
\begin{aligned}
\frac{dE_i(t,x)}{dt} &\le \left(\begin{aligned}
&-a_i p\gamma_p + 2^p \sum_{j=1}^{n}\left|b_{ij}\right| L_j + 2^p\gamma_p^p \sum_{j=1}^{n}\left|b_{ji}\right| L_i \\
&+2^p\left|l_i\right|\sum_{j=1}^{n}\left|b_{ij}\right| L_j + 2^p a_i\gamma_p^p \sum_{j=1}^{n}\left|q_i b_{ij}\right|^{p-1} L_j^{p-1} \\
&+2^p L_i^p\gamma_p^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\right)\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right)
\end{aligned}\right)\left|x_i(t)\right|^p \\
&+2^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right| L_i\right)\gamma_p^p\left|x_i(t-\tau)\right|^p \\
&+2^p L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1} a_j\right)\left|x_i(t-\sigma)\right|^p \\
&+2^p L_i^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\right)\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right)\left|x_i(t-\sigma)\right|^p \\
&+2^p L_i^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right)\left|x_i(t-\sigma)\right|^p \\
&+2^p L_i^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right)\gamma_p^p\left|x_i(t-\tau)\right|^p.
\end{aligned}
$$

Rearranging terms we have

$$
\begin{aligned}
\frac{dE_i(t,x)}{dt} &\le \left(\begin{aligned}
&-a_i\gamma_p\left(p - 2^p\gamma_p^{p-1}\sum_{j=1}^{n}\left|q_i b_{ij}\right|^{p-1} L_j^{p-1}\right) \\
&+2^p\gamma_p^p\sum_{j=1}^{n}\left|b_{ji}\right| L_i + 2^p\sum_{j=1}^{n}\left|b_{ij}\right| L_j\left(1+\left|l_i\right|\right) \\
&+2^p L_i^p\gamma_p^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\right)\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right)
\end{aligned}\right)\left|x_i(t)\right|^p \\
&+2^p\left[\begin{aligned}
&L_i^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left(1+\left|l_j\right|\right)\right)\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right) \\
&+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1} a_j\right)
\end{aligned}\right]\left|x_i(t-\sigma)\right|^p \\
&+2^p L_i\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\left[1 + L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_j b_{ji}\right|^{p-1}\right)\right]\gamma_p^p\left|x_i(t-\tau)\right|^p.
\end{aligned}
\qquad \textbf{(24)}
$$

Let $F(t,x)=\sum_{i=1}^{n}F_i\left(t,x\right)$, where

$$F_i\left(t,x\right)=2^p\left[\begin{array}{l}L_i^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left(1+\left|l_j\right|\right)\right)\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\\+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}a_j\right)\end{array}\right]\int_{t-\sigma}^{t}\left|x_i\left(s\right)\right|^p ds$$
$$+2^pL_i\gamma_p^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\left[1+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\right]\int_{t-\tau}^{t}\left|x_i\left(s\right)\right|^p ds. \quad \textbf{(25)}$$

Clearly $F_i\left(t,x\right)$ is positive definite, $F_i\left(t,0\right)=0$, $t\geq 0$ and is continuous on D_M. Now we verify that $F_i\left(t,x\right)$ is locally Lipschitzian on D_M^*. For any $(t,x),(t,\tilde{x})\in D_M^*=D_M-\left\{(t,0)/t\geq 0\right\}$, consider

$$F_i\left(t,x\right)-F_i\left(t,x\right)$$
$$=2^p\left[\begin{array}{l}L_i^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left(1+\left|l_j\right|\right)\right)\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\\+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}a_j\right)\end{array}\right]$$
$$\times\int_{t-\sigma}^{t}\left(\left|x_i\left(s\right)\right|^p-\left|\tilde{x}_i\left(s\right)\right|^p\right)ds$$
$$+2^pL_i\gamma_p^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\left[1+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\right]$$
$$\times\int_{t-\tau}^{t}\left(\left|x_i\left(s\right)\right|^p-\left|\tilde{x}_i\left(s\right)\right|^p\right)ds.$$

Using the inequality for any r_1,r_2,

$$\left|r_1^P-r_2^P\right|\leq p\left(r_1^{P-1}+r_2^{P-1}\right)\left|r_1-r_2\right|.$$

By applying the similar procedure for $E_i\left(t,x\right)$ is locally Lipschitzian on D_M^* as verified above, it can easily verify that $F_i\left(t,x\right)$ is also locally Lipschitzian on D_M^*.

Differentiate equation (25) w.r.t 't'

$$\frac{dF_i\left(t,x\right)}{dt}=2^p\left[\begin{array}{l}L_i^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left(1+\left|l_j\right|\right)\right)\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\\+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}a_j\right)\end{array}\right]$$
$$\times\left[\left|x_i\left(t\right)\right|^p-\left|x_i\left(t-\sigma\right)\right|^p\right]$$
$$+2^pL_i\gamma_p^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\left[1+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\right]$$
$$\times\left[\left|x_i\left(t\right)\right|^p-\left|x_i\left(t-\tau\right)\right|^p\right]. \quad \textbf{(26)}$$

Define

$$V\left(t,x\right)=E\left(t,x\right)+F\left(t,x\right)$$
$$=\sum_{i=1}^{n}E_i\left(t,x\right)+\sum_{i=1}^{n}F_i\left(t,x\right),t>0. \quad \textbf{(27)}$$

Clearly $V\left(t,x\right)$ is positive definite, $V\left(t,0\right)=0$, $t\geq 0$ and is continuous on D_M. Now we verify that $V\left(t,x\right)$ is locally Lipschitzian on D_M^*. $V\left(t,x\right)$ is locally Lipschitzian on D_M^*.Differentiating (27) w.r.t 't' and using (24) and (26)

$$\frac{dV\left(t,x\right)}{dt}$$
$$\leq\sum_{i=1}^{n}\left\{\left(\begin{array}{l}-a_i\gamma_p\left(p-2^p\gamma_p^{p-1}\sum_{j=1}^{n}\left|q_ib_{ij}\right|^{p-1}L_j^{p-1}\right)\\+2^p\sum_{j=1}^{n}\left|b_{ij}\right|L_j\left(1+\left|l_i\right|\right)+2^pL_i\gamma_p^p\sum_{j=1}^{n}\left|b_{ji}\right|\\+2^pL_i^p\gamma_p^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\right)\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\end{array}\right)\left|x_i\left(t\right)\right|^p\right.$$
$$+2^p\left[\begin{array}{l}L_i^p\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left(1+\left|l_j\right|\right)\right)\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\\+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}a_j\right)\end{array}\right]\left|x_i\left(t-\sigma\right)\right|^p$$
$$+2^pL_i\left(\sum_{j=1}^{n}\left|b_{ji}\right|\left|l_j\right|\right)\left[1+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\right]\left|x_i\left(t-\tau\right)\right|^p$$

$$+2^p\begin{bmatrix} L_i^p\left(\sum_{j=1}^n |b_{ji}|\left(1+|l_j|\right)\right)\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right) \\ +L_i^{p-1}\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1} a_j\right)\end{bmatrix}\left[|x_i(t)|^p - |x_i(t-\sigma)|^p\right]$$

$$+2^p L_i\gamma_p^p\left(\sum_{j=1}^n |b_{ji}||l_j|\right)\left[1+L_i^{p-1}\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right)\right]\left[|x_i(t)|^p - |x_i(t-\tau)|^p\right]\Bigg\}.$$

Simplifying

$$\frac{dV(t,x)}{dt} \le \sum_{i=1}^n \left\{ \begin{pmatrix} -a_i\gamma_p\left(p-2^p\gamma_p^{p-1}\sum_{j=1}^n |q_i b_{ij}|^{p-1} L_j^{p-1}\right) \\ +2^p\sum_{j=1}^n |b_{ij}| L_j\left(1+|l_i|\right)+2^p L_i\gamma_p^p\sum_{j=1}^n |b_{ji}| \\ +2^p L_i^p\gamma_p^p\left(\sum_{j=1}^n |b_{ji}|\right)\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right)\end{pmatrix}|x_i(t)|^p \right.$$

$$+2^p\begin{bmatrix} L_i^p\left(\sum_{j=1}^n |b_{ji}|\left(1+|l_j|\right)\right)\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right) \\ +L_i^{p-1}\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1} a_j\right)\end{bmatrix}|x_i(t)|^p$$

$$+2^p L_i\left(\sum_{j=1}^n |b_{ji}||l_j|\right)\left[1+L_i^{p-1}\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right)\right]|x_i(t)|^p .$$

Thus we have

$$\frac{dV(t,x)}{dt} \le -\sum_{i=1}^n \left\{ \begin{pmatrix} a_i\gamma_p\left(p-2^p\gamma_p^{p-1}\sum_{j=1}^n |q_i b_{ij}|^{p-1} L_j^{p-1}\right) \\ -2^p\sum_{j=1}^n |b_{ij}| L_j\left(1+|l_i|\right)+2^p L_i\gamma_p^p\sum_{j=1}^n |b_{ji}| \\ -2^p L_i^p\gamma_p^p\left(\sum_{j=1}^n |b_{ji}|\right)\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right) \\ -2^p\begin{bmatrix} L_i^p\left(\sum_{j=1}^n |b_{ji}|\left(1+|l_j|\right)\right)\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right) \\ +L_i^{p-1}\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1} a_j\right)\end{bmatrix} \\ -2^p L_i\left(\sum_{j=1}^n |b_{ji}||l_j|\right)\left[1+L_i^{p-1}\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right)\right]\end{pmatrix} |x_i(t)|^p \right. .$$

where

$$\mu = a_i\gamma_p\left(p-2^p\gamma_p^{p-1}\sum_{j=1}^n |q_i b_{ij}|^{p-1} L_j^{p-1}\right)$$
$$-2^p\sum_{j=1}^n |b_{ij}| L_j\left(1+|l_i|\right)+2^p L_i\gamma_p^p\sum_{j=1}^n |b_{ji}|$$
$$-2^p L_i^p\gamma_p^p\left(\sum_{j=1}^n |b_{ji}|\right)\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right)$$
$$-2^p\begin{bmatrix} L_i^p\left(\sum_{j=1}^n |b_{ji}|\left(1+|l_j|\right)\right)\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right) \\ +L_i^{p-1}\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1} a_j\right)\end{bmatrix}$$
$$-2^p L_i\left(\sum_{j=1}^n |b_{ji}||l_j|\right)\left[1+L_i^{p-1}\left(\sum_{j=1}^n |q_j b_{ji}|^{p-1}\right)\right]. \quad \textbf{(28)}$$

From (14) $\mu > 0$ then we have $\frac{dV(t,x)}{dt} \le 0$, $t > 0$.

Hence $V(t,x)$ is Liapunov functional for (4).Now we verify that $V'(t,x) \le -\lambda|x|^P$ on D_M, for some $\lambda > 0$, $p > 1$.

Since $\frac{dV(t,x)}{dt} \le -\mu|x(t)|^p$.

From lemma 4, $x = 0$ is L^P - stable for (4).

Example 1. Consider the network described by the system (4) with i=1, 2

$$a_1 = 7,\ a_2 = 8,\ \ q_1 = -\frac{1}{2},\ q_2 = \frac{1}{3}, l_1 = \frac{-1}{2},$$
$$l_2 = \frac{1}{5}, L_1 = \frac{1}{5},\ L_2 = \frac{1}{6},\ \left[b_{ij}\right] = \begin{bmatrix} 1 & 1 \\ 2 & 1\end{bmatrix},$$
$$\gamma_p = \max\{1, 2^{p-2}\}.$$

Further choose f_i, as follows for i=1, 2

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \tanh L_1 x_1 \\ \tanh L_2 x_1 \end{pmatrix}.$$

These parameters of the network satisfies conditions of Theorem 1

$$L_1\sum_{j=1}^{n}\left|b_{j1}\right|\frac{\left(\left|l_j\right|+1\right)}{a_j}=0.1029<1,$$

$$L_2\sum_{j=1}^{n}\left|b_{j2}\right|\frac{\left(\left|l_j\right|+1\right)}{a_j}=0.0607<1.$$

If $\eta=1/2,$

$$a_1\left(2-|q_1|\eta^{-1}\sum_{j=1}^{n}|b_{ij}|L_j\right)=11.4333$$
$$>\left(\sum_{j=1}^{n}|b_{1j}|L_j\eta^{-1}(1+|l_1|)+K_i\eta\sum_{j=1}^{n}|d_{j1}|\left(1+|m_j|+c_j|q_j|\right)\right.$$
$$\left.+K_1^2\left(\eta+\eta^{-1}\right)\left(\sum_{j=1}^{n}|d_{j1}||q_j|\right)\left(\sum_{j=1}^{n}|d_{j1}|\left(1+|m_j|\right)\right)\right)=2.8283,$$

$$a_1\left(2-|q_1|\eta^{-1}\sum_{j=1}^{n}|b_{ij}|L_j\right)=12.9778$$
$$>\left(\sum_{j=1}^{n}|b_{1j}|L_j\eta^{-1}(1+|l_1|)+K_i\eta\sum_{j=1}^{n}|d_{j1}|\left(1+|m_j|+c_j|q_j|\right)\right.$$
$$\left.+K_1^2\left(\eta+\eta^{-1}\right)\left(\sum_{j=1}^{n}|d_{j1}||q_j|\right)\left(\sum_{j=1}^{n}|d_{j1}|\left(1+|m_j|\right)\right)\right)=2.2551,$$

Thus conditions of theorem 2 are satisfied.
If $p=2.1,$ conditions of Theorem 3 are

$$A_i=a_i\gamma_p\left(p-2^p\gamma_p^{p-1}\sum_{j=1}^{n}\left|q_ib_{ij}\right|^{p-1}L_j^{p-1}\right),$$
$$B_i=2^p\sum_{j=1}^{n}|b_{ij}|L_j\left(1+|l_i|\right)+2^pL_i\gamma_p^p\sum_{j=1}^{n}|b_{ji}|$$
$$+2^pL_i^p\gamma_p^p\left(\sum_{j=1}^{n}|b_{ji}|\right)\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)$$
$$+2^p\left[\begin{array}{l}L_i^p\left(\sum_{j=1}^{n}|b_{ji}|\left(1+|l_j|\right)\right)\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\\+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}a_j\right)\end{array}\right]$$
$$+2^pL_i\left(\sum_{j=1}^{n}|b_{ji}||l_j|\right)\left[1+L_i^{p-1}\left(\sum_{j=1}^{n}\left|q_jb_{ji}\right|^{p-1}\right)\right].$$

$$A_1=13.2483\geq B_1=10.2520,$$
$$A_2=13.2483\geq B_2=10.2520.$$

Therefore conditions of Theorem 2 and 3 are satisfied, thus the equilibrium of the network is asymptotically stable and Lp-stable. If $\eta=5$, conditions of theorem 2 are not satisfied and if $p=1.5$

Then conditions of theorem3 are not satisfied. So p,η plays important role while satisfying the conditions of theorems 2 and 3 respectively.

Consider the model (1) with varying inputs, then the model become

$$\frac{d}{dt}\left[x_i(t)-q_i\sum_{j=1}^{n}b_{ij}f_j\left(x_j(t-\sigma)\right)-\frac{I_i(t)}{a_i}\right]$$
$$=-a_ix_i(t)+\sum_{j=1}^{n}b_{ij}f_j\left(x_j(t)\right)$$
$$+l_i\sum_{j=1}^{n}b_{ij}f_j\left(x_j(t-\tau)\right)+I_i(t),\ \ where\ \ I_i(t)=a_i\alpha_i(t). \quad \textbf{(29)}$$

Due to the fact that the input functions are time varying (no longer constants), the model (29) cannot have pre specified equilibrium patterns. In the applications of neural networks with optimization problems, state and output convergence of the network is basic constraint. some of the reasons to consider state and output convergence of the NNs with time-varying inputs are discussed in([5],[18]). Most of the research on these models (29) is focused on the output convergence analysis and it is stated that studying the state convergence of NNs with time varying inputs as model (29) in general is a difficult problem ([11],[18],[24],[,25]). In our investigation we study state convergence and provide analysis in restricted settings (A1-A3).

We assume that

(A1) the functions $f_i\ ,i=1,2,..,n,$ are globally Lipschitz continuous, monotone none decreasing activation functions and that is there exist $L_i>0$ such that

$$0\leq\frac{f_i(\mu)-f_i(\nu)}{\mu-\nu}\leq L_i, \quad \textbf{(30)}$$

for any $\mu,\nu\in R$ and $\mu\neq\nu.$ And also assume

for some $\xi>0,\|f_i'(x)\|\leq\xi,\ x\in R\ \ ,i=1,2,..,n$.

(A2) $I_i(t)$, are locally Lipschitz continuous that is if for every u in R there exist neighborhoods U_u such that I_i restricted to U_u respectively are Lipschitz continuous.

(A3) $I_i(t)$ satisfies the conditions

$$\lim_{t\to\infty}I_i(t)=\tilde{I}_i, \quad \textbf{(31)}$$

where $\tilde{I}_i$ are some constants. That is $\lim_{t\to\infty}I(t)=\tilde{I}.$

Theorem 4. Assume that (A1), (A2) and (A3) are satisfied and there exists a constant vector $x^*\in R^n$ such that

$$-a\,x^*+(1+l\,)B\,f(x^*)+\tilde{I}=0 \quad \textbf{(32)}$$

Then given any $x_0 \in R^n$ the system (29) has a unique solution $x(t;x_0)$ defined on $[0,\infty)$.
Proof of Theorem 4 is discussed in [5].

Now we obtain sufficient conditions for the equilibrium pattern to be globally exponentially stable. The equilibrium pattern x_i^* of (29) is said to be globally exponentially stable if there exist constants $\lambda > 0$ and $\eta \geq 1$ such that $\|x_i(t) - x_i^*\| \leq \|\phi_i - x_i^*\| \eta \; e^{-\lambda t}$ for any $t \geq 0$ We denote $\|\phi_i - x_i^*\| \leq \sup_{-\gamma \leq t \leq 0} \|\phi_i(t) - x_i^*\|$, where x_i^* the unique equilibrium of the system (29).

Theorem 5. Assume that the conditions (A1), (A2), and (A3) are satisfied and further suppose that there exist a positive constant $a = \min_{1 \leq i \leq n} \{a_i\}$ such that

$$a > \varepsilon = \sum_{i=1}^{n} \frac{L_i \left(\sum_{i=1}^{n} |b_{ji}| \left(1 + |l_j| + a_i |q_j|\right) \right)}{1 - L_i \left(\sum_{j=1}^{n} |b_{ji}| |q_j| \right)}. \qquad \textbf{(33)}$$

Then the equilibrium w^* of system (29) is globally exponentially stable.

Proof of Theorem 5 is discussed in [5].

IV. CONCLUSION AND REMARKS

In the present investigation, the authors have considered class of continuous- time hysteretic neuron model. Stability analysis is much desired for these systems from the point of view of the real world nature. We have obtained sufficient conditions for Input-output stability of a unique equilibrium. We have obtained asymptotic stability of the solutions of this system. The results are explicit in the sense that the criteria obtained are easily verifiable as they are expressed in terms of the parameters of the system. These models can be applied to a variety of real time applications such as the higher order hardware control systems can be replaced by this neural network for reducing the complexity, these neural networks can be desired and trained to filter out varying levels of noise interference in the channel and provide excellent data security and these neural networks can also be employed for image extraction in a varying noise interfering channels. In order to provide data security, a message (usually referred to as the plaintext) will be transformed by the sender into a random looking message (usually referred to as the ciphertext) by using an reversible mapping and transmitted to the receiver. However, during the transmission of the ciphertext in a noisy channel, the ciphertext gets altered disallowing the legitimate receiver to correctly get back the plaintext. To address this problem, all the ciphertexts(noise-free) can be stored as stable states of our network, so that whenever a noisy ciphertext is input to the network it converges after finite number of iterations to one of the stable states(the one to which its Hamming distance is the minimum) which will result in the correct plaintext after decryption.

The message to be transmitted will be stored as a binary image [3]. This image will then be encrypted using CDMA spreading technique, where PN-sequences will be generated using an LFSR whose connection polynomial is primitive. Noise of certain level will be added to the encrypted image and transmitted to the receiver. The receiver would then input this noisy pattern to the network (The network would store all the encrypted images of the messages that will be eventually transmitted). The pattern output by the network will then be decrypted using CDMA despreading technique.

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REFERENCES

[1] Aaron Strauss, Liapunov Functions and Lp Solutions of Differential Equations Source: Transactions of the American Mathematical Society, Vol. 119, No. 1 (Jul., 1965), pp. 37 -50 Published by: American Mathematical Society.

[2] C.Feng and R.Plamondon "Stability analysis of bidirectional associative memory networks with time delalys" – IEEE Transactions on Neural Networks, November 2003, Volume-14, Number 6,1560-1564.

[3] D. Gladis, P. Thangavel & A. K. Nagar (2011): Noise removal using hysteretic Hopfield tunnelling network in message transmission systems, International Journal of Computer Mathematics, 88:3, 650-660.

[4] G.Padmavathi, Kumar, P.V.S. and Bathul, S. (2013) 'Global exponential stability result for complex valued hysteretic neuron model', *Int. J. Machine Intelligence and Sensory Signal Processing*, Vol. 1, No. 1, pp.23–45.

[5] G.Padmavathi.P.V.Siva kumar,Exponential Stability of Hysteresis Neural Networks with Varying Inputs, 978-1-4673-5119-5/12/$31.00_c 2012 IEEE.

[6] G.W. Hoffman, A neural network model based on the analogy with the immune system, J.Theoret. Biol. 122 (1986) 33-67.

[7] H.Yanai, Y.Swada, Associative memory network composed of neurons with hysteretic property, Neural Networks, 3(1990) 223-228.

[8] Hirsch, M.W. (1989). Convergent activation dynamics in continuous time networks. Neural Networks, 2, 331-349

[9] J.P.Segundo and O.D.Martinez, Dynamic and static hysteresis in crayfish receptors, Biol.Cybern.vol.52, pp.291-296, 1985.

[10] K.Gopalsamy and Pingzhou Liu, Dynamics of hysteretic neuron, nonlinear analysis 8(2007), 375-398.

[11] Liu.D.,Hu,S.,& Wang.J. (2004), Global output convergence of a class of continuous-time recurrent neural networks with time-varying thresholds. IEEE Transactions on Circuits and Systems-II: Express Briefs, 51, 161-167.

[12] Mossaheb S. The circle criterion and the LP stability of feedback systems. SIAM J. Contr. Optim., Vol. 20, No. 1,1982, pp. 144–151.

[13] Pini Gurfil, Quantitative Lp stability analysis of a class of linear time varying feedback systems Int. J. Appl. Math. Comput. Sci., 2003, Vol. 13, No. 2, 179–184.

[14] Qiang Zhang, Xiaopeng wei and Jin Xu, A novel global exponential stability result for discrete-time cellular neural networks with variable delays, International journal of neural systems,vol.16,no.6(2006)467-472.

[15] R.L.Didday,"A model of visuomotor mechanisms in the frog optic tectum, Math.Biosci" vol.30, pp.169-180, 1976.

[16] Sandberg I.W. A frequency domain condition for the stability of feedback systems containing a single time varying nonlinear element. — Bell Syst. Tech. J., 1964, Vol. 43, pp. 1601–1608.

[17] Sandberg I.W. Some results on the theory of physical systems governed by nonlinear functional equations. — Bell Syst. Tech. J., 1965, Vol. 44, No. 5, pp. 871–898.[51]

[18] Sanquing Hu, Liu.D., on the global output convergence of a class of recurrent neural networks with time-varying inputs, Neural Networks, 18 (2005), 171-178..

[19] Sunil Bharitkav and Jerry M.Mendel ,The hysteretic hopfield neural network, “, IEEE Transactio ns on neural networks, Vol. 11. no. 4, July, 2000.

[20] Vidyasagar M. Nonlinear Systems Analysis, 2nd Ed.1993, New Jersey: Upper Saddle River.

[21] Y.P.Singh, The asymptotic behavior of solutions of linear third order differential equations .Proc.Amer.Math.Soc. 20(1969) 309-314.

[22] Y.Takefuji and K.C.Lee, A hysteresis binary neuron: A model suppressing the oscillatory behavior of neural dynamics, Biol.Cybern, vol.64, pp.353-356, 1991.

[23] Zames G. (1990): On input-output stability of time-varying nonlinear feedback systems-Part II: Conditions involving circles in the frequency plane and sector nonlinearities. — IEEE Trans. Automat. Contr., Vol. AC–11, No. 2, pp. 465–476.[72]

[24] Zhang Yi, Jian Cheng Lv, and Lei Zhang, Output convergence analysis for a class of delayed recurrent neural networks with time-varying inputs, IEEE Transactions on Systems, Man, and Cybernetics-PartB: Cybernetics, Vol. 36, No.1, February, 2006.

[25] Zhenyuan Guo, Lihong Huang, Global output convergence of a class of recurrent delayed neural networks with discontinuous neuron activations, Neural Process Lett (2009) 30:213-227, DOI 10.1007/s11063-009-9119-z.