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Article

Equitable Total Coloring of Various Snake Graphs

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Abstract: An Equitable Total Coloring (ETC) ensures that the sizes of all color classes, encompassing both edges and vertices differ by at most one, and the minimum number of colors required for equitable total coloring is known as the Equitable Total Chromatic Number (ETCN). This paper investigates the ETCN for... T_n, DT_n, DAT_n and DS_n .

Keywords: Total coloring, triangular snake, double alternate triangular snake, double triangular snake, diamond triangular snake graph and equitable total coloring.

INTRODUCTION

In this study, we consider basic finite graphs. Let G be a simple, undirected, linked graph. The sets of vertices and edges of G are represented by the conventional notations V(G) and E(G), respectively. A graph G's entire coloring gives its vertices and edges different colors so that no incident or neighboring elements have the same color. It is conjectured that the total chromatic number satisfies a specific inequality hold.: $\Delta(G)+1 \le \chi''(G) \le \Delta(G)+2$. In 1994, Fu [4] pioneered the concept of equitable total coloring and introduced the ETCN.

Later, Gong Kun et al. [3] established key results on the ETCN. of $W_n \vee K_n, F_m \vee K_n$ and $S_m \vee K_n$. Jayaraman and Muthuramakrishnan [5,6] proposed the ETCN for the middle graph, splitting graph, and total graph of paths, cycles, and the splitting graph of star graphs. Further contributions were made by Punitha and Jayaraman [7], who extended the analysis to middle graphs derived from snake graphs, particularly the triangular and quadrilateral variants. They computed the total chromatic numbers for these transformed structures and confirmed the validity of the Total Coloring Conjecture in each case. Their results provide a deeper understanding of how structural transformations influence equitable total coloring.

Vivik and Girija [8] determined the ETCN for wheel, gear, helm, and sunlet graphs. Gong Kun et.al [2] derived several findings regarding the ETCN for the graphs $W_n \vee K_n, F_n \vee K_n$ and $S_m \vee K_n$. Wang et.al [9] addressed the ETC for the graphs with a maximum degree of 3, while Zhang Zhong-fu [10] investigated the ETC of certain join graphs.

Girija and Vivik [11] conducted a detailed study on ETC for specific families of graphs, including double star graphs and fan graphs, which exhibit structural characteristics similar to those of snake graphs. They introduced the concept of the equitable total chromatic number, denoted as $\chi_{et}(G)$, and provided constructive proofs for coloring schemes that satisfy the equitable condition. Expanding on this foundation, Ramaprakash [12] focused on various types of snake graphs, including triangular, quadrilateral, and ladder configurations. The study examined their total chromatic numbers and introduced a classification system that divides snake graphs into Type I and Type II based on the bounds of their total chromatic numbers.

In a related direction, Mudrock et al. [13] introduced the concept of list ETC, a generalization of equitable total coloring applicable to more complex graph classes such as generalized theta graphs. Although not directly applied to snake graphs, the methods and conjectures proposed, including the List Equitable Total Coloring Conjecture, are relevant to snake-like structures due to their connectivity patterns based on paths..

Preliminaries

Definition 1. A Triangular snake T_n [1] is obtained from the $v_1, v_2, ..., v_n$ by joining v_i and v_{i+1} to a new vertex u_i for $i = 1, 2, 3, ..., n-1$.

Definition 2. A Double triangular snake DT_n consists of two triangular snakes that have a common path.

Definition 3. A Double Alternate Triangular Snake (DATn) comprises two alternate triangular snakes having a common path.

Definition 4. A Diamond Triangular Snake graph Dn is generated from a path through the replacement of each edge (K2) with two triangles (2C3).

RESULTS AND DISCUSSION

Theorem 1. For any $n \geq 3$, $\chi_e^{''}(T_n) = 5$.

Proof: Let $V(T_n) = \{u_\tau : 1 \leq \tau \leq n-1\} \cup \{v_\tau : 1 \leq \tau \leq n\}$ and

$$E(T_n) = \{e_\tau, e_\tau^{'}, f_\tau : 1 \leq \tau \leq n-1\} \text{, where } f_\tau = v_\tau v_{\tau+1}, e_\tau = u_\tau v_\tau,\ e_\tau^{'} = u_\tau v_{\tau+1}.$$

We divide the edge and vertex set of T_n into distinct partition as described below. Consider two cases

Case (i): Let n is even

$$T_1 = \{v_1, v_3, ..., v_{n-1}\} \cup \{e_1^{'}, e_3^{'}, ..., e_{n-3}^{'}\}$$
$$T_2 = \{v_2, v_4, ..., v_n\} \cup \{e_2^{'}, e_4^{'}, ..., e_{n-2}^{'}\}$$
$$T_3 = \{f_1, f_3, ..., f_{n-1}\} \cup \{u_1, u_3, ..., u_{n-1}\}$$
$$T_4 = \{f_2, f_4, ..., f_{n-2}\} \cup \{u_2, u_4, ..., u_{n-2}\} \cup \{e_{n-1}^{'}\}$$
$$T_5 = \{e_\tau : 1 \leq \tau \leq n-1\}$$

Case (ii): Suppose n is odd

$$T_1 = \{v_1, v_3, ..., v_n\} \cup \{e_1^{'}, e_3^{'}, ..., e_{n-2}^{'}\}$$
$$T_2 = \{v_2, v_4, ..., v_{n-1}\} \cup \{e_2^{'}, e_4^{'}, ..., e_n^{'}\}$$
$$T_3 = \{f_1, f_3, ..., f_{n-2}\} \cup \{u_1, u_3, ..., u_{n-2}\}$$
$$T_4 = \{f_2, f_4, ..., f_{n-1}\} \cup \{u_2, u_4, ..., u_{n-1}\}$$
$$T_5 = \{e_\tau : 1 \leq \tau \leq n-1\}$$

The edges and vertices of the graph T_n is colored with 5-colors. The color class of T_n are categorized as T_1, T_2, T_3, T_4, T_5 and these color classes are independent set. Thus the condition of ETC $\left\| T_i | - | T_j \right\| \leq 1$ for $i \neq j$ is satisfied. This implies that $\chi_e^{''}(T_n) \leq 5$ .Further, since $\Delta = 4$, we have $\chi_e^{''}(T_n) = \chi^{''}(T_n) \geq \Delta + 1 \geq 4 + 1 \geq 5$. Hence $\chi_e^{''}(T_n) = 5$.

Example 1. The graph T_6 and its equitable total coloring is shown in Figure 1.

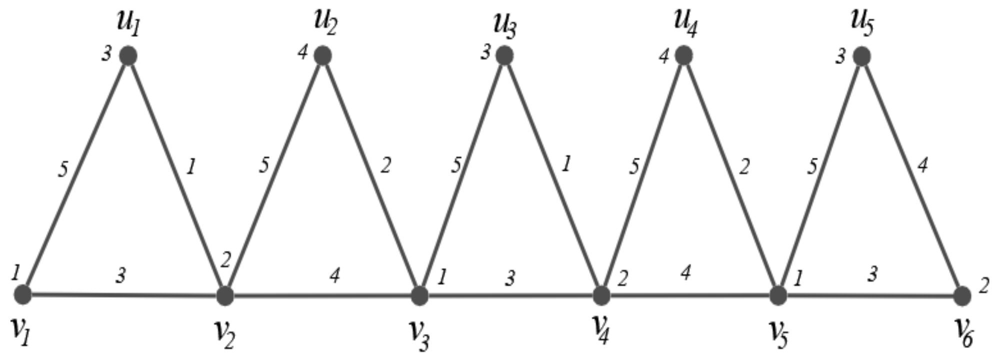

FIGURE 1. T_6 and its equitable total coloring

Example 2. The graph T_7 and its equitable total coloring is shown in Figure 2.

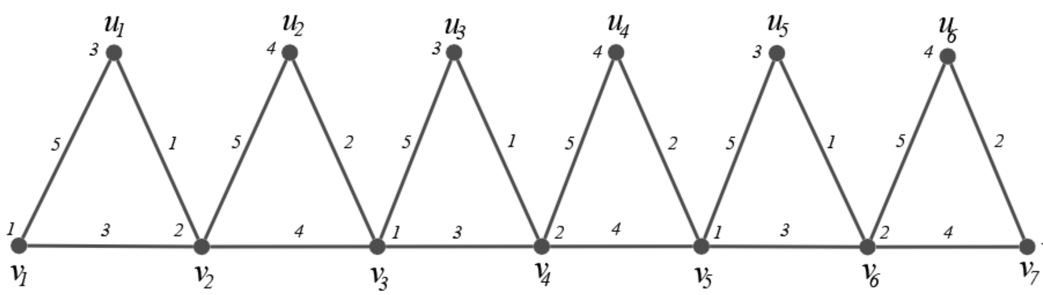

FIGURE 2. T_7 and its equitable total coloring

Theorem 2. Let DT_n be the double triangular snake graph, then $\chi_e^{''}(DT_n) = 7.$

Proof. $V(DT_n)) = \{u_\tau, v_\tau : 1 \le \tau \le n\} \cup \{u_\tau^{'} : 1 \le \tau \le n\}$

$E(DT_n)) = \left\{e_\tau, e_\tau^{'}, f_\tau, f_\tau^{'}, f_\tau^{''} : 1 \le \tau \le n-1\right\}$, where $e_\tau = u_\tau u_\tau^{'}$, $e_\tau^{'} = u_\tau u_{\tau+1}$, $f_\tau = u_\tau^{'} u_{\tau+1}^{'}$, $f_\tau^{'} = u_\tau^{'} v_\tau$, $f_\tau^{''} = v_\tau u_{\tau+1}^{'}$.

We divide the edge and vertex set of DT_n into distinct partition as described below.

Case (i): If $n = 1, 2, 3, \ldots.$ except $n \ne 7k, k = 1, 2, 3, \ldots..$

$$T_1 = \begin{cases} \{u_2, u_9, \ldots, u_{7\tau-5}\} \cup \{v_6, v_{13}, \ldots, v_{7\tau-1}\} \cup \{u_1^{'}, u_8^{'}, \ldots, u_{7\tau-6}^{'}\} \cup \{e_3, e_{10}, \ldots, e_{7\tau-4}\} \cup \\ \{e_6^{'}, e_{13}^{'}, \ldots, e_{7\tau-1}^{'}\} \cup \{f_5, f_{12}, \ldots, f_{7\tau-2}\} \cup \{f_4^{'}, f_9^{'}, \ldots, f_{7\tau-3}^{'}\} \cup \{f_1^{''}, f_8^{''}, \ldots, f_{7\tau-6}^{''}\} \end{cases}$$

$$T_2 = \begin{cases} \{u_3, u_{10}, \ldots, u_{7\tau-4}\} \cup \{v_7, v_{14}, \ldots, v_{7\tau}\} \cup \{u_2^{'}, u_9^{'}, \ldots, u_{7\tau-5}^{'}\} \cup \{e_4, e_{11}, \ldots, e_{7\tau-3}\} \cup \\ \{e_7^{'}, e_{14}^{'}, \ldots, e_{7\tau}^{'}\} \cup \{f_6, f_{13}, \ldots, f_{7\tau-1}\} \cup \{f_5^{'}, f_{12}^{'}, \ldots, f_{7\tau-2}^{'}\} \cup \{f_2^{''}, f_9^{''}, \ldots, f_{7\tau-5}^{''}\} \end{cases}$$

$$T_3 = \begin{cases} \{u_4, u_{11}, \ldots, u_{7\tau-3}\} \cup \{v_1, v_8, \ldots, v_{7\tau-6}\} \cup \{u_3^{'}, u_{10}^{'}, \ldots, u_{7\tau-4}^{'}\} \cup \{e_5, e_{12}, \ldots, e_{7\tau-2}\} \cup \\ \{e_8^{'}, e_{15}^{'}, \ldots, e_{7\tau+1}^{'}\} \cup \{f_7, f_{14}, \ldots, f_{7\tau}\} \cup \{f_6^{'}, f_{13}^{'}, \ldots, f_{7\tau-1}^{'}\} \cup \{f_3^{''}, f_{10}^{''}, \ldots, f_{7\tau-4}^{''}\} \end{cases}$$

$$T_4 = \begin{cases} \{u_5, u_{12}, ..., u_{7\tau-2}\} \cup \{v_2, v_9, ..., v_{7\tau-5}\} \cup \{u'_4, u'_{11}, ..., u'_{7\tau-3}\} \cup \{e_6, e_{13}, ..., e_{7\tau-1}\} \cup \\ \{e'_2, e'_9, ..., e'_{7\tau-5}\} \cup \{f_1, f_8, ..., f_{7\tau-6}\} \cup \{f'_7, f'_{14}, ..., f'_{7\tau}\} \cup \{f''_4, f''_{11}, ..., f''_{7\tau-3}\} \end{cases}$$

$$T_5 = \begin{cases} \{u_6, u_{13}, ..., u_{7\tau-1}\} \cup \{v_3, v_{10}, ..., v_{7\tau-4}\} \cup \{u'_5, u'_{12}, ..., u'_{7\tau-2}\} \cup \{e_7, e_{14}, ..., e_{7\tau}\} \cup \\ \{e'_3, e'_{10}, ..., e'_{7\tau-4}\} \cup \{f_2, f_9, ..., f_{7\tau-5}\} \cup \{f'_1, f'_8, ..., f'_{7\tau-6}\} \cup \{f''_5, f''_{12}, ..., f''_{7\tau-2}\} \end{cases}$$

$$T_6 = \begin{cases} \{u_7, u_{14}, ..., u_{7\tau}\} \cup \{v_4, v_{11}, ..., v_{7\tau-3}\} \cup \{u'_6, u'_{13}, ..., u'_{7\tau-1}\} \cup \{e_1, e_8, ..., e_{7\tau-6}\} \cup \\ \{e'_4, e'_{11}, ..., e'_{7\tau-3}\} \cup \{f_3, f_{10}, ..., f_{7\tau-4}\} \cup \{f'_2, f'_9, ..., f'_{7\tau-5}\} \cup \{f''_6, f''_{13}, ..., f''_{7\tau-1}\} \end{cases}$$

$$T_7 = \begin{cases} \{u_1, u_8, ..., u_{7\tau-6}\} \cup \{v_5, v_{12}, ..., v_{7\tau-2}\} \cup \{u'_7, u'_{14}, ..., u'_{7\tau}\} \cup \{e_2, e_9, ..., e_{7\tau-5}\} \cup \\ \{e'_5, e'_{12}, ..., e'_{7\tau-2}\} \cup \{f_4, f_{11}, ..., f_{7\tau-3}\} \cup \{f'_3, f'_{10}, ..., f'_{7\tau-4}\} \cup \{f''_7, f''_{14}, ..., f''_{7\tau}\} \end{cases}$$

Case (ii): When $n = 7k, k = 1,2,3,.....$

The proof is similar to case (i) except the only one vertex in the T_2 color class, so replace $v_{n-1} = 1$ to $v_{n-1} = 2$.

The edges and vertices of the graph DT_n is colored with 5-colors. The color class of DT_n are categorized as T_1, T_2, T_3, T_4, T_5 and these color classes are independent set. Thus the condition of ETC $\left\| T_i \right| - \left| T_j \right\| \le 1$ for $i \ne j$ is satisfied. This implies that $\chi''_e\left(DT_n\right) \le 6$ .Further, since $\Delta = 6$, we have $\chi''_e\left(DT_n\right) = \chi''\left(DT_n\right) \ge \Delta + 1 \ge 6 + 1 \ge 7$. Hence $\chi''_e\left(DT_n\right) = 7$.

Example 3. The graph DT_6 and its equitable total coloring is shown in Figure 3.

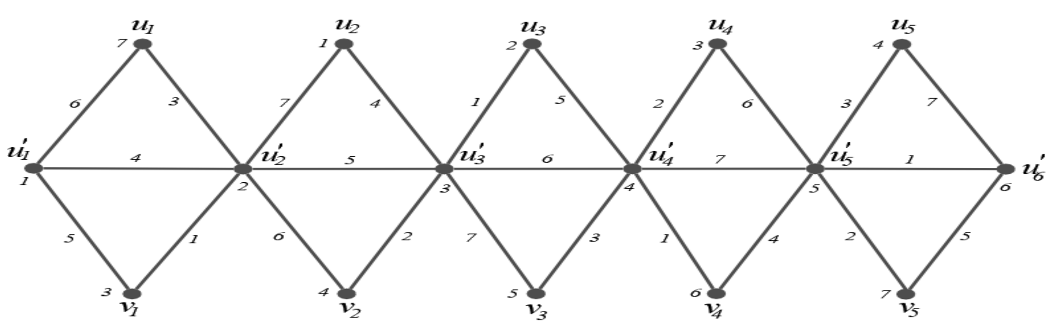

FIGURE 3. DT_6 and its equitable total coloring

Example 4. The graph DT_7 and its equitable total coloring is shown in Figure 4.

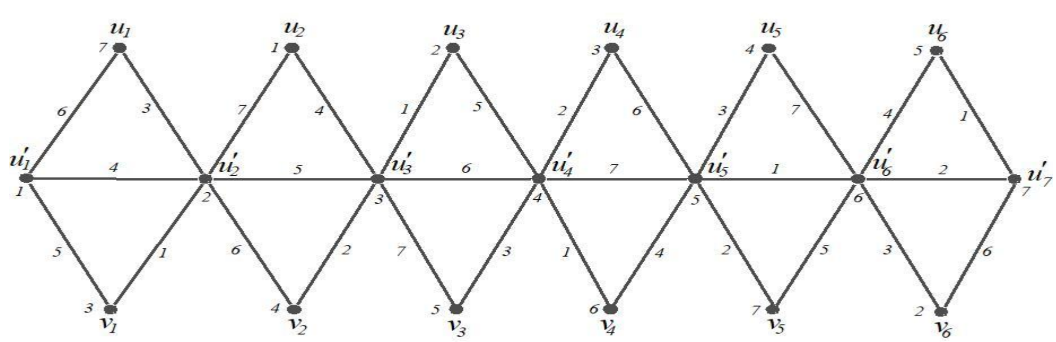

FIGURE 4. DT_7 and its equitable total coloring

Theorem 3. Let DAT_n be the double alternate triangular snake graph, then $\chi_e''(DT_n)=5$.

Proof: Let $V(DAT_n)=\left\{u_\tau, v_\tau : 1\le\tau\le\left\lfloor\frac{n}{2}\right\rfloor\right\}\cup\left\{u'_\tau : 1\le\tau\le n\right\}$

$E(DAT_n)=\left\{e_\tau, e'_\tau, f'_\tau, f''_\tau : 1\le\tau\le\left\lfloor\frac{n}{2}\right\rfloor\right\}\cup\left\{f_\tau : 1\le\tau\le n-1\right\}$, where $e_\tau=u_\tau u'_\tau$; $e'_\tau=u'_\tau u'_{\tau+1}$; $f_\tau=u'_\tau v'_{\tau+1}$, $f'_\tau=u'_\tau v_\tau$, $f''_\tau=v_\tau u'_{\tau+1}$,

We divide the vertex and edge set of DAT_n into distinct partition as described below.

$$T_1=\left\{u_1,u_2,\ldots,u_{\left\lfloor\frac{n}{2}\right\rfloor}\right\}\cup\left\{f_2,f_4,\ldots,f_{n-2}\right\}$$

$$T_2=\left\{v_1,v_2,\ldots,v_{\left\lfloor\frac{n}{2}\right\rfloor}\right\}\cup\left\{f_1,f_3,\ldots,f_{n-1}\right\}$$

$$T_3=\left\{e'_1,e'_2,\ldots,e'_{\left\lfloor\frac{n}{2}\right\rfloor}\right\}\cup\left\{f'_1,f'_2,\ldots,f'_{\left\lfloor\frac{n}{2}\right\rfloor}\right\}$$

$$T_4=\left\{u'_1,u'_3,\ldots,u'_{n-1,\ n\ even},u'_{n,\ n\ odd}\right\}\cup\left\{f''_1,f''_2,\ldots,f''_{\left\lfloor\frac{n}{2}\right\rfloor}\right\}$$

$$T_5=\left\{u'_2,u'_4,\ldots,u'_{n-1,\ n\ odd},u'_{n,\ n\ even}\right\}\cup\left\{e_1,e_2,\ldots,e_{\left\lfloor\frac{n}{2}\right\rfloor}\right\}$$

The edges and vertices of the graph DAT_n is colored with 5-colors. The color class of DAT_n are categorized as T_1,T_2,T_3,T_4,T_5 and these color classes are independent set. Thus the condition of ETC $\left\|T_i|-|T_j\right\|\le 1$ for $i\ne j$ is satisfied. This implies that $\chi_e''(DAT_n)\le 4$ .Further, since $\Delta=4$, we have $\chi_e''(DAT_n)=\chi''(DAT_n)\ge\Delta+1\ge 4+1\ge 5$. Hence $\chi_e''(DAT_n)=5$.

Example 5. The graph DAT_6 and its equitable total coloring is shown in Figure 5.

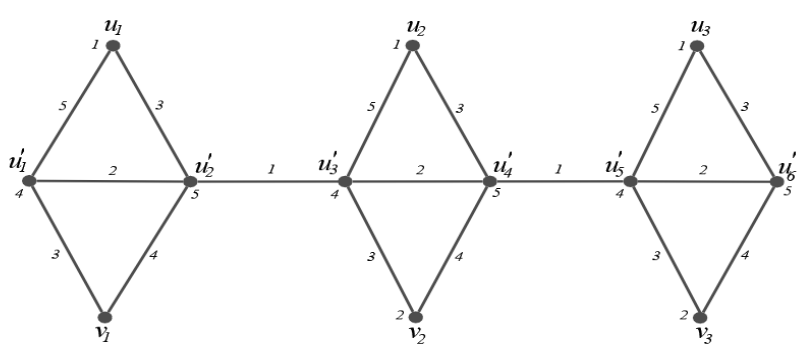

FIGURE 5. DAT_6 and its equitable total coloring

Example 6. The graph DAT_7 and its equitable total coloring is shown in Figure 6.

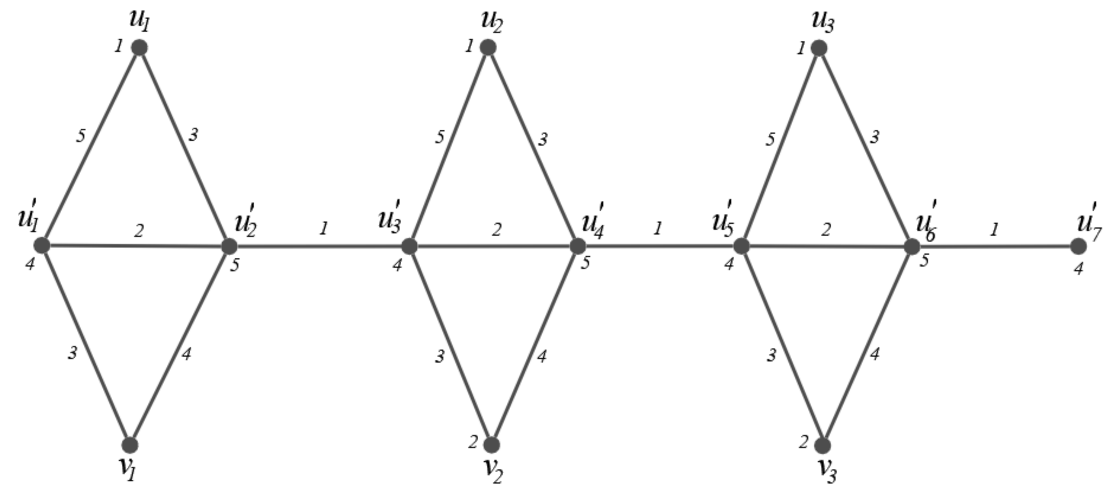

FIGURE 6. DAT_7 and its equitable total coloring

Theorem 4. Let D_n be the diamond snake graph then $\chi_e^{''}(D_n) = 5.$

Proof: $V(D_n)) = \{u_\tau, v_\tau : 1 \le \tau \le n-1\} \cup \{u_\tau' : 1 \le \tau \le n\}$ and

$$E(D_n)) = \left\{e_\tau, e_\tau', f_\tau, f_\tau' : 1 \le \tau \le n-1\right\}, \text{where } e_\tau = u_\tau u_\tau', e_\tau' = u_\tau u_{\tau+1}',\ f_\tau' = v_\tau u_{\tau+1}',\ f_\tau = u_\tau' v_\tau$$

We partition the edge and vertex set of D_n into distinct subsets as follows.

For $1 \le \tau \le n-1$.

$$T_1 = \begin{cases} \{u_5, u_{10}, ..., u_{5\tau}\} \cup \{v_5, v_{10}, ..., v_{5\tau}\} \cup \{u_3', u_8', ..., u_{5\tau-2}'\} \cup \{e_1, e_6, ..., e_{5\tau-4}\} \cup \\ \{e_4', e_9', ..., e_{5\tau-1}'\} \cup \{f_2, f_7, ..., f_{5\tau-3}\} \cup \{f_3', f_8', ..., f_{5\tau-2}'\} \end{cases}$$

$$T_2 = \begin{cases} \{u_1, u_6, ..., u_{5\tau-4}\} \cup \{v_1, v_6, ..., v_{5\tau-4}\} \cup \{u_4', u_9', ..., u_{5\tau-1}'\} \cup \{e_2, e_7, ..., e_{5\tau-3}\} \cup \\ \{e_5', e_{10}', ..., e_{5\tau}'\} \cup \{f_3, f_8, ..., f_{5\tau-2}\} \cup \{f_4', f_9', ..., f_{5\tau-1}'\} \end{cases}$$

$$T_3 = \begin{cases} \{u_2, u_7, ..., u_{5\tau-3}\} \cup \{v_2, v_7, ..., v_{5\tau-3}\} \cup \{u_5', u_{10}', ..., u_{5\tau}'\} \cup \{e_3, e_8, ..., e_{5\tau-2}\} \cup \\ \{e_1', e_6', ..., e_{5\tau-4}'\} \cup \{f_4, f_9, ..., f_{5\tau-1}\} \cup \{f_5', f_{10}', ..., f_{5\tau}'\} \end{cases}$$

$$T_4 = \begin{cases} \{u_3, u_8, ..., u_{5\tau-2}\} \cup \{v_3, v_8, ..., v_{5\tau-2}\} \cup \{u_1', u_6', ..., u_{5\tau-4}'\} \cup \{e_4, e_9, ..., e_{5\tau-1}\} \cup \\ \{e_2', e_7', ..., e_{5\tau-3}'\} \cup \{f_5, f_{10}, ..., f_{5\tau}\} \cup \{f_1', f_6', ..., f_{5\tau-4}'\} \end{cases}$$

$$T_5 = \begin{cases} \{u_4, u_9, ..., u_{5\tau-1}\} \cup \{v_4, v_9, ..., v_{5\tau-1}\} \cup \{u_2', u_7', ..., u_{5\tau-3}'\} \cup \{e_5, e_{10}, ..., e_{5\tau}\} \cup \\ \{e_3', e_8', ..., e_{5\tau-2}'\} \cup \{f_1, f_6, ..., f_{5\tau-4}\} \cup \{f_2', f_7', ..., f_{5\tau-3}'\} \end{cases}$$

The graph's edges and vertices are colored using 5 distinct colors. The color class of D_n are categorized as T_1, T_2, T_3, T_4, T_5 and these color classes are independent set. Thus the condition of ETC $\left\| T_i \right| - \left| T_j \right\| \le 1$ for $i \ne j$ is satisfied. This proves that $\chi_e^{"}\left(D_n\right) \le 4$ .Further, since $\Delta = 4$, we have $\chi_e^{"}\left(D_n\right) = \chi^{"}\left(D_n\right) \ge \Delta + 1 \ge 4 + 1 \ge 5$ Hence $\chi_e^{"}\left(D_n\right) = 5.$

Example 7. The graph and its equitable total coloring are depicted in Figure 7.

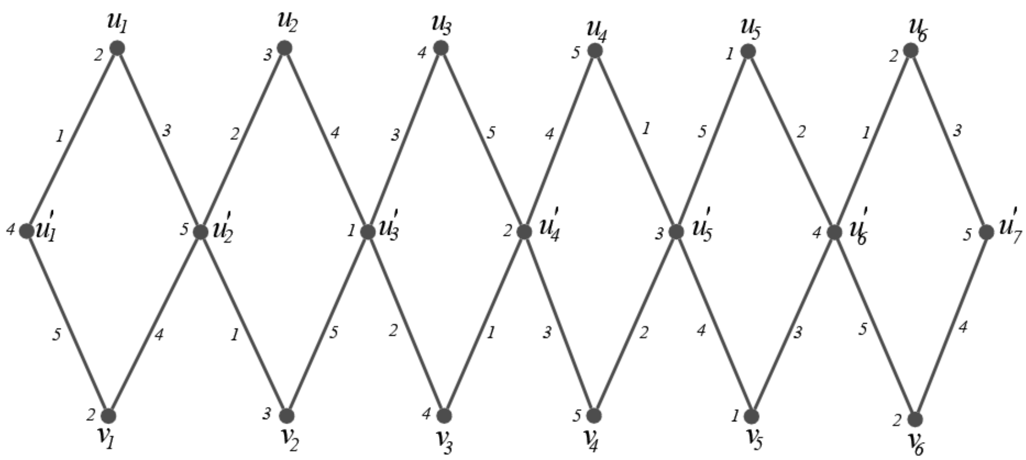

FIGURE 7. Equitable Total Coloring Illustration

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