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Article

A Methodological Framework for Converting ARCH and GARCH Financial Time Series Models to a Kink Regression Model

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Abstract: Autoregressive Conditional Heteroskedasticity (ARCH) models and their generalized form, GARCH, have been widely used for several decades as an important tool for modeling the volatility of financial time series. However, applying them to data with structural changes leads to biased estimates and unreliable results. This study aims to present a comprehensive theoretical framework for the mathematical transformation from these two models to the kink regression model developed by Hansen (2017), through a detailed mathematical derivation that demonstrates how the transition occurs mathematically. The proposed theoretical framework operates in several stages, beginning with reformulating the conditional mean function to accommodate breakpoints, followed by addressing conditional heteroskedasticity through two alternative approaches, then defining the unknown inflection points using a grid search, and concluding with a test of the significance of the transformation using the Wald statistic. The framework was applied to monthly data from the Iraq Stock Exchange spanning from February 2015 to December 2025, with a sample size of 131 observations, The T-statistic () was found to be (=47.638), indicating the presence of two inflection points with significant economic implications: the first point occurred at an exchange rate of 1,211 dinars per U.S. dollar (prior to the devaluation of the dinar against the dollar in December 2020), and the second point was at an oil price of $61.75 per barrel (i.e., close to the approximate break-even price for the Iraqi budget). This study offers a new contribution to the field of financial econometrics by establishing a mathematical and methodological approach that links modern time-series models with the threshold regression framework.
Keywords: ARCH and GARCH models, kink regression, structural changes, financial time series, Wald test, emerging economies, Iraq Stock Exchange.

1. Introduction

Financial time series modeling has undergone a phase of theoretical and applied development since the early 1980s. These changes were unprecedented and began with the work of Engle (1982), who established the foundations of the Autoregressive Conditional Heteroskedasticity (ARCH) model. This was followed by Bollerslev (1986) introduced his generalized (GARCH) model, which has become a highly significant and influential tool in financial econometrics. This trajectory culminated in Engle being awarded the Nobel Prize in Economics in 2003 for the major advances he made in understanding the mechanisms of volatility in global financial markets.

However, the practical application of these models in emerging financial markets reveals a fundamental difficulty whose impact has not been adequately addressed in the mainstream literature. Unlike their developed counterparts, these markets occasionally experience political and institutional events that cause structural breaks in data behavior. Over the past ten years, the Iraqi economy has experienced multiple events, the most significant of which was the devaluation of the Iraqi dinar in December 2020 from 1,190 Iraqi dinars to 1,460 Iraqi dinars per U.S. dollar, and the ensuing structural impacts on various economic sectors, including the securities market.

The main problem with the methodology lies in the fact that the ARCH and GARCH time series models assume stability in the overall structure of the data in the conditional mean; this assumption loses its validity in data characterized by sudden jumps in economic relationships. As Engle (2001) noted during a preliminary review of these models that their practical application to data that contradicts their fundamental assumptions will lead to biased estimates and inaccurate results, and consequently, decisions based on them will be inaccurate and unreliable. Here, we must ask how researchers can benefit from previous, information-rich theoretical studies and findings on ARCH and GARCH models, as well as how they can address emerging economies that simultaneously exhibit structural changes.

The kink regression model, introduced by Hansen (2017), derives its strength and significance from its ability to detect such changes in data structure, as it is a model in which the slope of the regression function changes at a kink point while the continuity of the function is maintained; this property allows for the modeling of various phenomena in which the dependent variable is affected by sudden changes in the independent variables, even though the level of the function remains constant at the kink point. This model has been successfully applied in various fields, including the impact of tax policies on taxpayer behavior (Saez, 2010) and the effect of unemployment benefits on the job search period (Landais, 2015).

This study aims to address gaps in the existing literature, which consist of the absence of a unified methodology that illustrates the transition from ARCH and GARCH models to the kink regression model. This shift does not diminish the role of time-series models in analysis but rather expands their scope to include various types of changes in the data that could not previously be clearly addressed, The main significance and major contribution of this study focus on three main areas. The first area is a presentation of the mathematical model and its constraints, with a detailed, sequential explanation of the limitations faced by the ARCH and GARCH models in the presence of non-stationary data; The second pillar is the mathematical derivation, which explains how to transform these models into the mathematical formulation of kink regression and the accompanying proofs. The third pillar is the practical application to data from the Iraqi Stock Exchange for the period from February 2015 to December 2025.

This study consists of six parts, with the explanation provided above constituting the first part; the second part reviews previous studies related to the topic, through which the main research gap between the various studies is identified; the third part presents the theoretical and mathematical framework for the mathematical models; the fourth part is dedicated to the mathematical mechanism used in the transformation; and the final section is a practical application to data from the Iraq Stock Exchange for the period under study. The study concludes with the sixth section, which summarizes the findings and outlines future research directions.

2. Literature Review

The studies and literature relevant to this study follow overlapping yet distinct paths, beginning with developments in financial time-series models and the accompanying notable advancements in regression models specifically, threshold regression and kink regression models as they seek to link all the paths they follow. Engle's (1982) is considered the turning point and initial starting point in modeling volatility in financial data, as he introduced the ARCH model, which makes the conditional variance a function of the squared values of past errors, according to his mathematical formula ($\sigma_t^2 = \beta_0 + \sum \beta_i u_{t-i}^2$) This model addressed a significant gap in financial data analysis by accounting for volatility clustering a phenomenon where it is difficult to capture the stability of variance in linear models. However, from a scientific standpoint, the ARCH(q) model encountered the problem of relying on high orders to represent long-term memory, which led Bollerslev (1986) to adopt the generalized GARCH(p,q) formula, as it allows for the use of past values of autocovariance and reduces the number of required parameters, thereby facilitating their estimation while maintaining the best representation.

In the same vein, Nelson (1991) concluded that negative indicators and news events increase market volatility more than positive indicators of the same magnitude, and he developed the EGARCH model, which captures these asymmetric effects by using the logarithm of the conditional variance. Furthermore, Engle and Lee (1999) expanded this framework to include the analysis of volatility across two horizons: the first being long-term and the second short-term. This, in turn, allows for a distinction between volatility inherent in the data structure and volatility that is not significant. However, despite the importance of these models over the past four decades, they remain limited and constrained by a single key assumption: the stability of the data structure over time. Yet this assumption cannot be accepted unconditionally in emerging financial markets. Engle (2001) noted this limitation of the models in his review of ARCH/GARCH models, pointing out that any deviation from the underlying assumptions leads to bias in the estimates and yields inaccurate and unreliable statistical results.

Studies examining regression models with thresholds were further developed by Tong (1983, 1990), who established the fundamental assumptions for threshold autoregressive (TAR) models, as well as their estimation and inference methods. Hamilton (1989) contributed a new perspective to this literature through Markov-switching models, which allow for transitions between different states based on an underlying Markov chain a feature that enables a more realistic representation of economic cycles. On the theoretical front, Chan (1993) established that the convergence results obtained from the least squares method in threshold models are fundamentally based on the convergence rate of the threshold estimator itself, and that this convergence is of order (n^{-1}) rather than ($n^{-1/2}$) an important result reflecting the nonstandard nature of the estimation equation, Chan and Tsay (1998) further investigated the asymptotic properties of this convergence rate of the least-squares estimator in models with a continuous threshold.

The important study conducted by Hansen (2017) served as an important convergence point between the two previous approaches by introducing a kink regression model with an unknown threshold, as it combines the representational flexibility of threshold models with the mathematical properties of continuous functions. A key advantage of this model is that the function remains continuous at the kink point; however, the slope changes abruptly at the kink point, which corresponds to certain economic phenomena in which sensitivity changes without a jump in the relationship. Hansen derived the properties of the various parameter estimates and showed that the estimate of (r) (the kink point) converges at a rate of ($n^{-1/3}$), while the estimates of the slope parameters ($\boldsymbol{\beta}$) converge at the standard rate of ($n^{-1/2}$). He also developed the Wald test to detect the significance of a kink point by introducing critical tables that take into account the non-standard nature of the statistical distribution.

The kink regression model has been successfully and clearly applied in various applied economic studies, including Card, Lee, Pei, and Weber (2012), who used the model within what they termed the Generalized Regression Kink Design to identify the causal effects of public policies. Saez (2010) used the same model to study taxpayer behavior at kink points in marginal tax rates, revealing the phenomenon of "bunching," which is particularly pronounced near these kink points. Subsequently, Landais (2015) contributed to evaluating the effects of unemployment compensation programs on job search duration. Furthermore, Ganong and Jäger (2014) employed a permutation test to assess the accuracy of identifying the presence of kink points.

As for attempts to establish a link between financial time series models and the kink regression model, previous studies are relatively few in number despite the theoretical and practical importance of this topic. Tibprasorn, Maneejuk, and Sriboonchitta (2017) came close to establishing this link by applying the kink regression model to cross-sectional data and estimating it using the general maximum entropy method. Furthermore, Tarkhamtham, Yamaka, and Sriboonchitta (2018) made a significant contribution by estimating the parameters of the kink regression model using Tsallis's entropy measure; they further expanded on this in the study by Tarkhamtham and Yamaka (2019) by examining higher-order entropy estimates. Pele and Mazurencu-Marinescu-Pele (2019) drew attention to the importance of higher-order entropy in risk forecasting, which in turn reinforced the significant and strong link between the entropy literature and the financial time series literature.

In recent studies, however, there have been numerous attempts to integrate traditional models with machine learning techniques, in line with developments in this field, as seen in the study by Han et al. (2024) which, in turn, presented a hybrid model that combines GARCH, CEEMDAN analysis, and convolutional neural networks to improve the accuracy of financial time series forecasting. However, this hybrid model lacks the statistical reliability that characterizes traditional models, which in turn makes it difficult to interpret the economic relationship between the obtained results difficult. Ultimately, the greatest challenge remains clearly evident in the absence of a comprehensive methodological framework that clarifies the mathematical transition mechanism from ARCH and GARCH models to the kink regression model in a reliable manner, as well as the clarity of the conditions indicating that this transition is valid and the inferential statistical tools demonstrating the model's significance. This study seeks to bridge the gap between existing studies by presenting the proposed methodology in the following sections, focusing on the mathematical properties and then working to prove them directly.

3. Mathematical Theoretical Framework

This section of the study will present the mathematical formulations of the models examined in this study in a precise and detailed manner, in addition to outlining the key statistical properties of each model and its conditions of validity, this review will begin with the ARCH model and an overview of its basic structure, followed by a discussion of the generalized (GARCH) model, then the kink regression model, and finally an explanation of the theoretical aspects of the Wald test.

3.1 ARCH Model

An ARCH model of order (**q**) is based on the assumption that the conditional variance of the random error term at period **t** (time point) depends significantly on the squares of the error terms from previous periods; that is, if we denote the dependent variable (the response variable) by ($\mathbf{Y}_t$) and the vector of independent variables by ($\mathbf{X}_t$), and the vector of slope parameters by (***β***), then the model will take the following form:

$$\mathrm{Y}_t = \acute{X}_t\beta + u_t \qquad {u_t}/{F_{t-1}} \sim N(0, \sigma_t^2) \qquad \text{... (1)}$$

$$\sigma_t^2 = \beta_0 + \sum_{i=1}^{q} \beta_i u_{t-i}^2 \qquad \text{... (2)}$$

Where F_{t-1} represents the set of information at time step (t-1), To ensure that the variance is positive at every time step, $\beta_0 > 0$ and $\beta_i \geq 0$ for all i =1, 2, 3, …, q. We can write the unconditional expectation of the variance, assuming it is stationary, as follows:

$$E(\sigma_t^2) = \frac{\beta_0}{1 - \sum_{i=1}^{q} \beta_i} \qquad \text{... (3)}$$

From Equation (3), it can be seen that the unconditional variance is positive and is finite if and only if the sum of the parameters is less than one; this condition is essential for the model's heterogeneous stability, and its absence implies an increase in variance over time. We can also prove that the unconditional variance of the squared residuals (u_t^2) follows an ARMA model, which can be written as follows:

$$u_t^2 = \beta_0 + \sum_{i=1}^{q} \beta_i u_{t-i}^2 + v_t \qquad \text{... (4)}$$

where $v_t = \sigma_t^2(z_t^2 - 1)$ and $z_t = {u_t}/{\sigma_t} \sim N(0,1)$, This formulation provides an important insight into the essence of the ARCH model, which is essentially an ARMA model based on the residuals' squares, and this facilitates parameter estimation using the Conditional Maximum Likelihood method. The necessary condition for estimating these parameters is satisfied when the regularity conditions are met, as they involve the mathematical derivation of the likelihood function and the absence of singularity in the Fisher information.

3.2 The GARCH Model and the Long-Memory Property

The GARCH model, introduced by Bollerslev (1986), addresses the shortcomings of the ARCH model by using higher orders to represent long memory. The variance equation in the GARCH(p,q) model is written as follows:

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \beta_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \qquad \text{... (5)}$$

The simple and commonly used mathematical formula is GARCH(1,1), which is written as follows:

$$\sigma_t^2 = \omega + \beta_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \qquad \text{... (6)}$$

By substituting σ_{t-1}^2 into Equation (6), we obtain an infinite expansion of the variance:

$$\sigma_t^2 = \frac{\omega}{1 - \beta_1} + \beta_0 \sum_{i=0}^{\infty} \beta_1^i u_{t-1-i}^2 \qquad \text{... (7)}$$

From Equation (7), it appears that the GARCH(1,1) model is practically equivalent to an ARCH model of infinite order with parameters that decay geometrically at a rate of β_1. This explains its ability to capture long-memory effects with a limited number of parameters, since its heteroskedasticity stability requires that ($\beta_0 + \beta_1 < 1$), and the unconditional variance is calculated according to the following formula:

$$E(\sigma_t^2) = \frac{\omega}{1 - \beta_0 - \beta_1} \qquad \text{... (8)}$$

It should be noted that the closer the sum of ($\beta_0 + \beta_1$) is to one, the more it indicates that the impact of shocks on variance persists over long periods; this is the characteristic of long-memory. However, if the sum equals one, this IGARCH (Integrated GARCH) model, in which the unconditional variance is undefined, indicating that the stationarity condition has been violated.

3.3 Kink Regression Model

Hansen (2017) introduced the kink regression model as a flexible alternative that allows the slope to change at an unknown point while maintaining the continuity of the function at that point. The general mathematical form of this model for a single independent variable is written as follows:

$$y_t = \beta_0 + \beta_1 (x_t - r)^- + \beta_1 (x_t - r)^+ + u_t \qquad \text{... (9)}$$

Here, the negative and positive regions are expressed using the following formulas:

$$(x_t - r)^- = min(x_t - r, 0), \qquad (x_t - r)^+ = max(x_t - r, 0) \qquad \text{... (10)}$$

We can prove the continuity of the function when ($x_t = r$) and here we observe that the limits on the right and left sides are equal:

$$\lim_{x \to r^-} f(x) = \lim_{x \to r^+} f(x) = \beta_0 \qquad \text{... (11)}$$

The right-hand and left-hand derivatives take two different values, namely (β_1^+) and (β_1^-) respectively; this justifies calling r the kink point. The difference between (β_1^+) and (β_1^-), represents the magnitude of the kink and is considered the primary parameter through which its statistical significance is tested. The model can also be extended to the case of multiple independent variables (explanatory variables), with each variable having its own kink point. and if there are (K) independent variables, the extended formula will take the following form:

$$y_t = \beta_0 + \sum_{k=1}^{K} \left[\beta_k (x_{k,t} - r_k)^- + \beta_1 (x_{k,t} - r_k)^+ \right] + u_t \qquad \text{... (12)}$$

Furthermore, the nonlinear nature of the model will become apparent here, as the design matrix itself relies on unknown kink points; this is the key feature that distinguishes this model from standard linear regression models.

3.4 Properties of the Convergent Kink Regression Model Estimator

The kink regression model is estimated using the nonlinear least squares method by minimizing the sum of the squares of the residuals with respect to the model parameters, including the value of ($\boldsymbol{r}$). Hansen (2017) proved that the estimates obtained possess very important convergence properties, as the estimate of the kink vector ($\hat{\boldsymbol{r}}$) converges to the true value ($\boldsymbol{r_0}$) at a rate of ($n^{-1/3}$), that is:

$$n^{\frac{1}{3}} (\hat{\boldsymbol{r}} - \boldsymbol{r_0}) \to \gamma \qquad as \quad n \to \infty \qquad (13)$$

where (γ) is a random vector that follows a complex distribution, as it depends on the derivative of the random process surrounding the original kink points; as for the convergence rate ($n^{-1/3}$), it is slower than the standard rate ($n^{-1/2}$) This reflects the non-smooth nature of the estimation function at kink points. Despite this slow convergence of ($\hat{\boldsymbol{r}}$), the estimates of the slope parameters ($\widehat{\boldsymbol{\theta}}$) will converge at the standard rate ($n^{-1/2}$) and will asymptotically follow a multinomial distribution:

$$\sqrt{n} \left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta_0} \right) \to dN(0, v_\theta) \qquad (14)$$

Since $\boldsymbol{v_\theta}$ represents the covariance matrix, which is estimated using standard methods, this result allows for the construction of confidence intervals and the testing of standard hypotheses for the kink parameters.

3.5 The Wald Test for Detecting a Kink

Hansen (2017) developed a specialized statistical test for detecting the significance of a kink; it tests the null hypothesis (H_0: $\beta_1^- = \beta_1^+$), which implies the absence of a kink, against the following alternative hypothesis (H_1: $\beta_1^- \neq \beta_1^+$), which implies the presence of a kink. The test statistic is calculated using the following formula:

$$T_N = n . \frac{SSE_R - SSE_U}{SSE_U} \qquad \text{... (15)}$$

Since SSE_R represents the sum of the squares of the residuals under the linear model, and SSE_U is calculated under the nonlinear model (which has a kink point), and n represents the sample size and is subject to the null hypothesis, as it is distributed according to the T_N statistic, which approximates a non-standard distribution based on a Gaussian process:

$$T_N \rightarrow d_{r \in \gamma} sup[W(\acute{r})W(r)] \qquad \dots (16)$$

Since W(r) is a Gaussian random process dependent on (r) and (γ), it represents the set of permissible values for the kink point. This result is unusual because the asymptotic distribution is not the usual chi-square (χ^2) distribution, but rather the supremum of a Gaussian process. Hansen (2017) have compiled critical tables based on Monte Carlo simulations to estimate the critical values at various significance levels.

4. The Mathematical Framework for the Transformation

This section constitutes a significant contribution to this study, as it presents the detailed mathematical steps for transitioning from a time series model of the ARCH or GARCH form to a kink regression model. The idea is based on a fundamental observation: the assumption of structural stability in the conditional mean of financial time series breaks down in the face of data exhibiting kinks, requiring the replacement of the linear form of the conditional mean with a kinked form that accommodates these kinks. The transformation will proceed through four consecutive stages, which we will present below along with the supporting mathematical derivations for each.

4.1 Framework Assumptions and Validity Conditions

Before reviewing the steps of the mathematical transformation, it is necessary to establish a set of assumptions upon which the proposed framework for this study is built. The first assumption is that the original time-series data exhibit a notable clustering of fluctuations, which justifies the use of an ARCH or GARCH model in the initial analysis of the data. This assumption can be tested using the Ljung-Box test on the residuals from a preliminary linear regression model:

$$Q(m) = n(n+2)\sum_{k=1}^{m} \frac{\hat{\rho}_k^2(u^2)}{n-k} \qquad \dots (17)$$

Since $\hat{\rho}_k^2(u^2)$ represents the kth-order autocorrelation coefficient of the residual squares; if the value of Q(m) exceeds the critical value of the chi-square distribution (χ_m^2) at a specified significance level, we reject the hypothesis that the residual squares are independent, which indicates the dynamic nature of ARCH/GARCH models.

The second assumption posits the existence of external explanatory variables $x_{1,t}, x_{2,t}, x_{3,t}, \dots, x_{k,t}$ that are related to the dependent variable (y_t) according to a relationship based on kink points, and the third assumption posits the existence of economic or political evidence indicating the likelihood of kink points; this third assumption is considered the most important from an applied perspective, as it determines the necessity of transformation that is, if the data under study are stationary, there is no reason for transformation, and thus the traditional GARCH model remains a suitable choice for the data. However, if the data is non-stationary that is, if there are actual structural changes in the data then relying on traditional models will lead to biased estimates.

4.2 Step 1: Reformulating the conditional mean

The first step involves reformulating the conditional mean of the time series model from its linear form to a new refractive form (containing a kink). Instead of the linear form in Equation (1), we propose a refractive form that incorporates K as a potential kink point:

$$E(\mathrm{y}_t/F_{t-1}) = \beta_0 + \sum_{k=1}^{K}\left[\beta_k\left(x_{k,t}-r_k\right)^- + \beta_1\left(x_{k,t}-r_k\right)^+\right] \qquad \dots (18)$$

This transformation involves a significant expansion of the parameter space, as the number of parameters increases from (K+1) in the linear form to (3k+1) in the form with a kink point, (by adding 2k slope parameters and K inflection points). However, this expansion is necessary to capture the change in the sensitivity of (y_t) to changes in each of the independent variables at its inflection point (i.e., its own kink point). From an interpretive standpoint, the parameters (β_1^-) and (β_1^+) each represent the sensitivity of (y_t) to changes in $(x_{k,t})$ before the kink point (r_k) and after the kink point, respectively, where the difference between (β_1^-) and (β_1^+) serves as a measure of the strength of the inflection at that point (the kink point).

4.3 Step 2: Addressing Conditional Heteroskedasticity

The final step is to address conditional heteroskedasticity, which occurs after reformulating the conditional mean. It is very important to distinguish between two possible causes of clustering in the actual data: The first cause is the dynamic nature of real GARCH models, which reflect the autocorrelation of variance, and the second cause is a spurious clustering resulting from ignoring kinks in the conditional mean. The clustering observed in the original case may be due to the latter cause that is, it disappears after incorporating into the model. To address this issue, this study proposes two different methodologies. The first methodology assumes that the variance remains constant after the transformation process is complete, meaning that:

$$\text{Var}(u_t \mid F_{t-1}) = \sigma^2 \qquad \text{(constant)} \qquad \ldots (19)$$

This assumption is justified by the fact that a large portion of the data's variability is attributable to the structural changes that the kink regression model is designed to detect. After applying the model transformation, the phenomenon of residual variability clustering may disappear, which justifies a return to the assumption of constant variance. We can test the validity of this assumption by reapplying the Ljung -Box test on the residuals from the transformed model.

The second methodology presents a hybrid model because it combines kink regression for the conditional mean and GARCH for the conditional variance. This methodology is suitable for cases where volatility remains clustered even after structural adjustment, and this hybrid model is expressed as follows:

$$y_t = \beta_0 + \sum_{k=1}^{K} \left[\beta_k (x_{k,t} - r_k)^- + \beta_1 (x_{k,t} - r_k)^+\right] + u_t \qquad \ldots (20)$$

$$u_t = \sigma_t z_t, \quad where \quad z_t \sim \text{iid}(0,1) \qquad \ldots (21)$$

$$\sigma_t^2 = \omega + \beta_0 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \qquad \ldots (22)$$

Since Equations (20) and (22) are used to form the Kink-GARCH(1,1) model, which is estimated using the conditional maximum likelihood method, and the log-likelihood function is given by the following equation:

$$\text{L}(\theta, r, \psi) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\left[\ln(\sigma_t^2) + \frac{u_t^2}{\sigma_t^2}\right] \qquad \ldots (23)$$

where ($\boldsymbol{\theta}$) represents the vector of slope parameters and the constant term, **r** represents the vector of kink points, and $\boldsymbol{\psi} = (\boldsymbol{\omega}, \boldsymbol{\beta_0}, \boldsymbol{\beta_1})$ represents the GARCH model parameters; the GARCH model parameters are estimated using the iterative BHHH method after determining appropriate initial values for them (Berndt et al., 1974).

4.4 Step Three: Identification of Kink Points

The process of discovering and identifying kink points (r_k) is considered one of the most technically complex aspects of the transformation process. This is in contrast to the slope parameters, which are estimated using linear methods once the kink points are known and (r_k) into the model in a nonlinear manner; they are non-derivable, and the estimation equation can be formulated as a two-level optimization problem as follows:

$$\hat{r} = \underset{\gamma}{\text{argmin}}\left[\min_{\theta}\sum_{t=1}^{n}(y_t - X_t(r)'\theta)^2\right] \qquad \ldots (24)$$

First, we will find the optimal value of ($\boldsymbol{\theta}$) subject to **r** (which is a standard linear equation with a closed-form solution), and then we will find the value of r that minimizes the resulting function. The interior-point solution will be as follows:

$$\hat{\theta}(r) = [X(r)'X(r)]^{-1}X(r)'y \qquad \ldots (25)$$

Hansen (2017) stipulates that, to ensure model identification, the region (before and after the kink point) must contain a sufficient number of observations. He also suggests, for practical purposes, that the proportion of observations in each region should not be less than 15% of the sample size under study, and that this condition requires the network analysis to be limited to the domain $[Q_{0.15}(x_k), Q_{0.85}(x_k)]$, where $Q_\alpha(x_k)$ represents the region between two percentiles β of the distribution of xk, namely (15%) and (85%).

4.5 Step 4: Verifying the Significance of the Transformation

The transformation process is considered complete if it is statistically significant based on the Wald test according to Equation (15). Under the null hypothesis i.e., the absence of a kink T_n follows the non-standard distribution defined in Equation (16), and the calculated value is compared with the critical values in the Hansen (2017) or by estimating them using the bootstrap method.

5. Practical Application

To demonstrate the practical feasibility of the proposed methodology used in this study, this section will present a practical application using data from the Iraq Stock Exchange for the period spanning from February 2015 to December 2025, amounting to (131 monthly views). This data was selected for two important reasons. First, the Iraqi economy experienced major events during this period that led to clear structural changes, the most significant of which was the devaluation of the Iraqi dinar in December 2020 by approximately 23%. The second consideration is that the Iraqi economy is a rentier economy, as it relies entirely on oil revenues; this makes financial indicators highly sensitive to any fluctuations in global oil prices.

5.1 Study Variables and Mathematical Formulation

The practical application of this study involves three main variables: the dependent variable is the logarithm of the Iraq Stock Exchange **(ISX)** general index, $\mathbf{ln(ISX_t)}$; the independent variables are the logarithm of the exchange rate of the Iraqi dinar against the U.S. dollar $\mathbf{ln(ER_t)}$ and the logarithm of the price of Brent crude oil in dollars per barrel $\mathbf{ln(Oil_t)}$. The three-time series were subjected to standard stationarity tests using the Augmented Dickey-Fuller method:

$$\Delta y_t = \rho\, y_{t-1} + \sum_{i=1}^{p} \phi_i\, \Delta y_{t-i} + \nu_t \qquad \dots (26)$$

Since all the chains are of first-order I(1) symmetry, the kink model equation takes the following empirical form:

$$\ln(\mathrm{ISX}_t) = \beta_0 + \beta_1(\ln(\mathrm{ER}_t) - r_1)^- + \beta_1(\ln(\mathrm{ER}_t) - r_1)^+ + \beta_2(\ln(\mathrm{Oil}_t) - r_2)^- + \beta_2(\ln(\mathrm{Oil}_t) - r_2)^+ + \boldsymbol{u_t} \qquad \dots \mathbf{(27)}$$

5.2 Wald Test Results and Parameter Estimates

The Wald test was applied to the actual study data to detect the presence of significant kink points. This test yielded a statistical value (T_n=47.638), which is more than five times the critical value at the 1% significance level (9.21 according to Hansen 2017). This is conclusive evidence of the presence of two significant kink points, which provides full support for completing the transformation process and estimating the kink model.

The network analysis identified two kink points, each of which has economic significance. The kink point in the exchange rate relationship was estimated at (r_1=7.0956) in logarithmic units, which corresponds to an exchange rate of approximately 1,211 dinars per U.S. dollar. This value matches with remarkable precision the exchange rate that prevailed prior to the official devaluation of the Iraqi dinar in December 2020. Similarly, the kink point in the oil price relationship was estimated at (r_2= 4.1231) in logarithmic units, which is equivalent to 61.75$ per barrel a value close to the approximate break-even price for the Iraqi national budget.

Table (1): Parameters of the kink regression model estimated after transformation

Parameter	Symbol	Estimate	Standard error	t-value
constant	β_0	6.4357	0.0424	151.79***

Exchange rate trend (before kink)	β_1^-	22.4858	7.2356	3.108***
Exchange rate trend (after kink)	β_1^+	1.0952	0.2640	4.149***
Oil price trend (before kink)	β_2^-	0.3747	0.1035	3.620***
Oil price trend (after kink)	β_2^+	–0.7193	0.1423	-5.055***

*** Statistically significant at the 1% level.

5.3 Economic Interpretation of the Results

The results obtained, as shown in Table (1), reveal very significant economic dynamics. In the lower exchange rate regime (i.e., before the dinar devaluation), the index's sensitivity to exchange rate changes is high, reaching a value of ($\hat{\beta}_1^-$= 22.49). This indicates that fluctuations in the official exchange rate even if slight during that period had a significant and strong impact on the index. In contrast, under the upper regime (i.e., after the dinar devaluation), this sensitivity decreases significantly, reaching ($\hat{\beta}_1^+$= 1.10) This suggests that the market has adapted to the new regime and has become less responsive to its additional fluctuations.

Furthermore, the most important point to note is that the relationship between the oil price and the index reverses from a positive relationship before the kink point ($\hat{\beta}_2^-$=0.37) to a negative relationship after the kink point ($\hat{\beta}_2^+$=-0.72). This asymmetric change requires a multidimensional economic explanation. One possible explanation is that oil prices rising above the break-even price generate domestic inflationary pressure, which reduces consumers' purchasing power and affects the profits of listed companies. The second explanation points to the possibility that investors turn to alternative investments, such as real estate and gold, during the oil boom, thereby reducing inflows into the relatively narrow financial market.

6. Conclusions and Prospects for Future Research

This study has presented a comprehensive analytical roadmap for the transformation from the ARCH and GARCH financial time-series models to a kink regression model, based on a highly rigorous mathematical foundation and an illustrative application to real-world data. The four transformation steps revealed a clear path that begins with reformulating the moving average, through handling conditional variance and identifying kink points, and ending with verifying the significance of the transformation via the Wald test. This study represents an important contribution toward bridging an existing gap in the literature regarding the methodological link between the frameworks of financial time series models and kink regression models.

The practical application of this method to data from the Iraq Stock Exchange has demonstrated the effectiveness of the proposed approach as well as the feasibility of its use, as the Wald test yielded a statistical value (T_n) of 47.638 a value that significantly exceeds the critical thresholds (kink points) at any significance level. The analysis revealed two kink points, each of which has significant economic implications: the first corresponds to the devaluation of the Iraqi dinar in December 2020, and the second corresponds to the approximate break-even point for the Iraqi budget. One of the most important aspects to note in the results is the shift (reversal) in the relationship between the oil price and the index from a positive to a negative relationship upon crossing the kink point, which represents the break-even price, this finding warrants further in-depth analysis in subsequent studies, as this research opens up numerous avenues for future research. The first of these avenues involves expanding the proposed framework to include multiple-kink regression models that is, models with more than one kink point in a single independent variable. The second avenue involves involves developing a dynamic version of the model that allows the kink points themselves to change over time i.e. ($r_k = r_k(t)$The third area involves applying this framework to other Arab, regional, and global markets such as the Saudi, Jordanian, Turkish, and U.S. markets, among others thereby enabling informative comparative studies.

In conclusion, we hope that this study has contributed to building a methodological bridge between two frameworks, each of which is rich in econometric literature, even though they have been separate from one another despite their natural connections. The field remains open for researchers to further develop and expand this bridge to enhance our understanding of financial markets in emerging economies .

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