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Article

On the Connected Distance–k Domination Numbers of Standard Graphs

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Abstract: This paper investigates connected distance-k dominating sets in graphs, in which every vertex is within distance k from a connected dominating set. The connected distance- k domination number, denoted $\gamma_{c\leq k}(G)$, represents the minimal cardinality of a minimal connected distance- k dominating set. The difference between the connected distance- k domination number and the standard distance- k domination number is expressed as $\tau_{c\leq k}(G) = \gamma_{c\leq k}(G) - \gamma_{\leq k}(G)$, which is referred to as the connected distance- k domination transition number. Various constraints on the connected distance- k domination number and its transition number have been established. For a number of common graphs, the precise values of these recently created parameters are calculated, and their connections to other dominating parameters are also investigated. Furthermore, results of the Nordhaus-Gaddum type are given for these new parameters.

Keywords: Dominating set, Connected dominating set, connected distance -k domination number, distance -k dominating set, and transition number.

1. Introduction

Only simple, finite, connected, and undirected graphs were examined in the present study. Let n and m represent the graph G's size and order, respectively. We referred to it using [2]. Let the maximum (minimum) degree be represented by $\Delta(G)(\delta(G))$. The expression $\lceil x\rceil(\lfloor x\rfloor)$ represents the least (largest) integer larger (less) than or equal to x The highest cardinality among the independent set of vertices of G is known β_0(G). If a vertex v of G is next to a pendent vertex, it is referred to as a support. An internal vertex is any vertex with a degree greater than one.

When every vertex in V - D is adjacent to at least one vertex of D, the non-empty subset $D \subseteq V$ of vertices in a graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ is referred to as a dominant set. The symbol $\gamma(\mathrm{G})$ represents the domination number of G, which is the smallest cardinality of a minimal dominating set. A dominating set's maximal cardinality is its higher domination number (G) [3,5]. Ore [6] was the first to use the word "domination".

Numerous authors have defined and examined different kinds of domination parameters, and Haynes et al. [2] appendix had a list of over 80 domination models [4]. The notion of linked domination in graphs was first presented by Sampathkumar and Walikar [7]. If the induced sub-graph $\langle D\rangle$ is connected, then the dominating set $D \subseteq V$ of G is referred to as a connected dominating set. The connected domination number of G, represented as $\gamma_S(G)$, is the smallest cardinality of a minimal connected dominating set of G. If $\langle D\rangle$ has no isolated vertices, then a dominating set $D \subseteq V$ of G is referred to as a whole dominating set of G. The total domination number of G, represented by $\gamma_t(G)$, is the smallest cardinality of a total dominating set of G [9,10].

If every vertex in $\mathrm{V} - \mathrm{D}$ is within distance -k of at least one vertex in D , then a non-empty set $D \subseteq V$ of vertices in a graph $\mathrm{G} = (\mathrm{VE})$ is a distance -k dominant set. The lowest cardinality of a minimal distance -k dominating set in G is equal to the distance -k domination number $\gamma \leq \mathrm{k}(\mathrm{G})$ of G [2, 8].

Definition 1.1

If the induced sub-graph $\langle D\rangle$ is connected, then a distance -k dominating set $D \subseteq V$ in a graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ is a connected distance -k dominating set. The smallest cardinality of the minimal connected distance -k dominating set is the connected distance -k dominance number $\gamma_{c\leq \mathrm{k}}(G)$ of G. The greatest cardinality of an aconnected distance - k dominating set is the higher connected distance - k dominance number $\Gamma_{c\leq \mathrm{k}}(G)$ of G.

Example 1.2

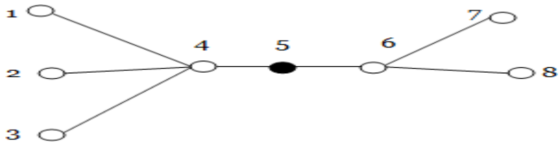

Figure.1

Here $\gamma(G) = 2, \gamma_c(G) = 3$, when $k = 2, \gamma_{\leq 2}(G) = 1, \gamma_{c\leq 2}(G) = 1, \Gamma_{c\leq 2}(G) = 9$.

2. Standard Graphs and Their Values of $\gamma_{c\leq 2}(G)$ for k=2.

The connected distance-2 domination number $\gamma_{c\leq 2}(G)$ for some standard graphs is given as follows.

2.1 Observations

For the path graph P_n,

$$\gamma_{c\leq 2}(P_n) = \begin{cases} n-4, & n \geq 5 \\ 1, & n < 5 \end{cases}$$

For the cycle graph C_n,

$$\gamma_{c\leq 2}(C_n) = \begin{cases} n-4, & n \geq 5 \\ 1, & n < 5 \end{cases}$$

For the grid graph $P_i \times P_j$, with $i = 2,3$ and $j \geq 3$,

$$\gamma_{c\leq 2}(P_i \times P_j) = j - 2$$

For the lollipop graph $L_{m,n}$ and all m,

$$\gamma_{c\leq 2}(L_{m,n}) = \begin{cases} n-2, & n \geq 3 \\ 1, & n < 3 \end{cases}$$

2.2 Observations

For the following standard graphs, the connected distance- k domination number satisfies:

For the wheel graph $W_n, n \geq 3$,

$$\gamma_{c\leq k}(W_n) = 1.$$

For the friendship graph $F_n, n \geq 2$,

$$\gamma_{c\leq k}(F_n) = 1$$

For the star graph $K_{1,m}, m \geq 1$,

$$\gamma_{c\leq k}(K_{1,m}) = 1.$$

For the complete graph $K_n, n \geq 3$,

$$\gamma_{c\leq k}(K_n) = 1$$

For the book graph $B_n, n \geq 3$,

$$\gamma_{c\leq k}(B_n) = 1$$

For the helm graph $H_n, n \geq 3$,

$$\gamma_{c\leq k}(H_n) = 1$$

For the n-barbell graph, $n \geq 3$,

$$\gamma_{c\leq k}(n\text{-barbell}) = 1$$

For the ladder rung graph L_n,

$$\gamma_{c\leq k}(L_n) = 0$$

For the complete bipartite graph $K_{n,m}, n, m \geq 1$,

$$\gamma_{c\leq k}(K_{n,m}) = 1$$

For every graph G,

$$\gamma_{c\leq k}(K_n + G) = 1$$

where $K_n + G$ denotes the join of K_n with G.

3. Graph theoretical parameters of $\gamma_{c\leq}(G)$

Theorem 3.1

For any star graph $K_{1,n}$, $\gamma_{\leq k}(K_{1,n}) = \gamma_{c\leq k}(K_{1,n}) = 1$.

Proof.

In a star graph, one central vertex is adjacent to all other vertices. Hence, the domination number is $\gamma(K_{1,n}) = 1$. Since the maximum distance in a star graph is one, it follows that $\gamma_{\leq k}(K_{1,n}) = 1$. Moreover, singleton sets are connected [7], which yields

$$\gamma_{\leq k}(K_{1,n}) = \gamma_{c\leq k}(K_{1,n}) = 1.$$

Theorem 3.2

If G is a book graph, complete bipartite graph or a helm graph, then

$$\gamma_{\leq k}(G) = \gamma_{c\leq k}(G) = 1$$

Theorem 3.3

If G is a a wheel graph, a complete graph or a friendship graph then

$$\gamma(G) = \gamma_c(G) = \gamma_{\leq k}(G) = \gamma_{c\leq k}(G) = 1$$

Theorem 3.4

For a book or a complete bipartite graph, the following holds:

$$\gamma_{c\leq}(G) = \gamma_c(G) - 1.$$

Theorem 3.5

For the helm graph H_n,

$$\gamma_{c\leq k}(H_n) = \gamma_c(H_n) - (n-1),$$

where n denotes the suffix index of H_n.

Theorem 3.6

For a complete bipartite or a book graph, one has

$$\gamma_{c\leq 2}(G) = \gamma(G) - 1.$$

Theorem 3.7

In the helm graph H_n,

$$\gamma_{c\leq k}(H_n) = \gamma(H_n) - (n-1).$$

Proposition 3.8

If G is a star graph, complete or wheel graph with $n \geq 2$, then

$$\gamma_{c\leq k}(G) = \frac{n}{\Delta(G)+1}$$

where $\Delta(G)$ is the maximum degree of G.

Proposition 3.9

Let G be a cycle, a path, a helm a book or a complete bipartite graph with $n \geq 2$. Then

$$\gamma_{c\leq k}(G) = \left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor$$

4. Connected Distance-2 Domination Number and its bounds

Proposition 4.1

For any connected graph G, $\gamma_{\leq k}(G) \leq \gamma_{c\leq k}(G)$

Proof.

Every connected distance- k dominating set of G is also a distance- k dominating set of G. Hence,

$$\gamma_{\leq k}(G) \leq \gamma_{c\leq k}(G).$$

Proposition 4.2

For any connected graph G,

$$\gamma_{c\leq k}(G) \leq \gamma_c(G).$$

Proof.

Every connected dominating set of G is a connected distance- k dominating set of G. Thus,

$$\gamma_{c\leq k}(G) \leq \gamma_c(G).$$

Proposition 4.3

For any connected graph G,

$$\gamma_{c\leq k}(G) \leq \gamma(G),$$

if and only if G is not a path or a cycle.

Proposition 4.4

Let G be a connected graph and H a connected spanning subgraph of G. Then, every connected distance$k$ dominating set of H is also a connected distance- k dominating set of G. Consequently,

$$\gamma_{c\leq 2}(G) \leq \gamma_{c\leq 2}(H)$$

Hedetniemi and Laskar [2] established the inequality for the connected domination number:

$$\gamma_c(G) \leq n - \Delta(G),$$

where $\Delta(G)$ is the maximum degree of G.

Theorem 4.5

For any connected graph G,

$$\gamma_{c\leq k}(G) \leq n - \Delta(G).$$

Proof.

From [4], $\gamma_c(G) \leq n - \Delta(G)$. By Proposition 4.2, $\gamma_{c\leq k}(G) \leq \gamma_c(G)$. Hence,

$$\gamma_{c\leq k}(G) \leq n - \Delta(G).$$

Theorem 4.6

For any connected graph G,

$$\gamma_{c\leq k}(G) \leq 2\beta_0 - 1, \gamma_{c\leq k}(G) \leq 3\gamma(G) - 2$$

where β_0 is the independence number of G.

Proof.

From [2], we have $\gamma_c(G) \leq 2\beta_0 - 1$ and $\gamma_c(G) \leq 3\gamma(G) - 2$. By Proposition 4.2, $\gamma_{c\leq k}(G) \leq \gamma_c(G)$. Therefore,

$$\gamma_{c\leq k}(G) \leq 2\beta_0 - 1 \text{ and } \gamma_{c\leq k}(G) \leq 3\gamma(G) - 2.$$

Corollary 4.7

For any connected graph G,

$$\gamma_{c\leq k}(G) \leq 3\gamma_{\leq k}(G) - 2.$$

Bo and Liu [2] showed that the irredundance number satisfies

$$\gamma_c(G) \leq 3\mathrm{ir}(G) - 2$$

where $\mathrm{ir}(G)$ denotes the irredundance number.

Theorem 4.8

For any connected graph G,

$$\gamma_{c\leq k}(G) \leq 3\mathrm{ir}(G) - 1.$$

Proof.

From [2], $\gamma_c(G) \leq 3\mathrm{ir}(G) - 2$. By Proposition 4.2, $\gamma_{c\leq k}(G) \leq \gamma_c(G)$. Hence,

$$\gamma_{c\leq k}(G) \leq 3\mathrm{ir}(G) - 1.$$

Theorem 4.9

For any connected graph G of order n :

(i) $\gamma_{c\leq k}(G) + \gamma_{\leq k}(G) \leq n$,

(ii) $\gamma_{c\leq k}(G) + \gamma(G) \leq n$.

Theorem 4.10

If G is a connected graph of order $n \geq 5$, then:

(i) $\gamma_{c\leq 2}(G) = n - 4$ if and only if $G \cong P_n$ or C_n,

(ii) $1 \leq \gamma_{c\leq k}(G) \leq n - 4$.

Nordhaus-Gaddum Type Results

For any graph G and its complement $\bar{G}$ (without isolated vertices):

(i) $2 \leq \gamma_{c\leq k}(G) + \gamma_{c\leq k}(\bar{G}) \leq 2(n-4)$,

(ii) $1 \leq \gamma_{c\leq k}(G) \cdot \gamma_{c\leq k}(\bar{G}) \leq (n-4)^2$.

5. Graphs with Equality between Connected- k Domination and Distance- k Domination Numbers

Theorem 5.1

Let G be a connected graph, and let D denote a minimum connected distance-2 dominating set of G. If $\gamma_{\leq 2}(G) = \gamma_{c\leq 2}(G)$, then the maximum degree of the induced subgraph $\langle D \rangle$ satisfies

$$\Delta(\langle D \rangle) < \Delta(G)$$

where $\Delta(G)$ represents the maximum degree of a vertex in G[1].

Proof.

Assume, that $\Delta(\langle D \rangle) = \Delta(G)$. Let $v \in D$ be a vertex such that $\deg_{\langle D \rangle}(v) = \Delta(G)$ as its contradiction. Removing v from $D - \{\}$ yields a distance-2 dominating set of cardinality $\gamma_{\leq 2}(G) - 1$, contradicting the minimality of D. Hence, the claimed inequality holds.

Theorem 5.2

Let T_n be a tree of order $n \geq 3$. For $k = 2$, the equality $\gamma_{\leq k}(T_n) = \gamma_{c\leq k}(T_n)$ holds iff each internal vertex of T_n has a support vertex within distance two [1].

Proof.

Let T_n be a tree with $n \geq 3$. Let D represent the set of all internal vertices within a distance of two from the support of the tree, which forms the unique minimum connected distance-2 dominating set in T_n. We know that $\gamma_{\leq 2}(T_n) = \gamma_{c\leq 2}(T_n)$.

Suppose there exists an internal vertex v that does not have a support within distance two. Then, $D - \{v\}$ would form a distance-2 dominating set with a cardinality of $\gamma_{\leq 2}(T_n) - 1$, which leads to a contradiction.

Conversely, if every internal vertex of T_n has a support within distance two, then it follows that $\gamma_{\leq 2}(T_n) \geq |D|$.

Thus, we conclude that $\gamma_{\leq 2}(T_n) = \gamma_c \leq 2(T_n)$.

Theorem 5.3

Let G be a connected cubic graph of order $n \leq 8$.

Proof.

If $\gamma_{\leq 2}(G) = \gamma_c \leq 2(G)$, then G must be isomorphic to one of the following: $K_4, \overline{C_6}, K_{3,3}$, or one of the graphs G_1, G_2, G_3, or G_4, as depicted in Figure 2.

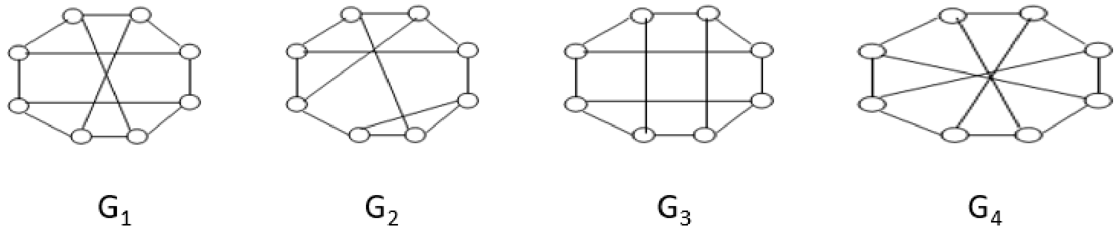

Figure 2. Cubic graphs

For the graphs $G = K_4, \overline{C_6}, K_{3,3}$, it holds that

$$\gamma_{\leq 2}(G) = \gamma_{c\leq 2}(G) = 1$$

On the other hand, for $G = G_1, G_2, G_3$, or G_4, we have

$$\gamma_{\leq 2}(G) = \gamma_{c\leq 2}(G) = 2$$

Theorem 5.4

For any path graph P_n, the connected distance- k domination number of its centralization satisfies

$$\gamma_{\leq k}\big(C(P_n)\big) = 1 = \gamma_{c\leq k}\big(C(P_n)\big)$$

Proof.

Let P_n be a path of length $n-1$ with vertices $v_1, v_2, \ldots, v_n$. During the centralization process, let u_i denote the subdivision vertex corresponding to edge $v_i v_{i+1}$ for $1 \leq i \leq n-1$, and let the subdivided edges be $v_i u_i = e_i$ and $u_i v_{i+1} = e_i'$.

In the centralized graph $C(P_n)$, each vertex v_i is adjacent to all vertices except its immediate neighbors v_{i-1} and v_{i+1} (for $1 \leq i \leq n$). However, every vertex v_i is at distance two from v_{i-1} and v_{i+1}, and all subdivision vertices u_i lie within distance two of v_i. Therefore, a single vertex v_i forms a distance-2 dominating set of $C(P_n)$, implying

$$\gamma_{\leq 2}\big(C(P_n)\big) = 1 = \gamma_{c\leq 2}\big(C(P_n)\big)$$

This argument generalizes for distance k as well.

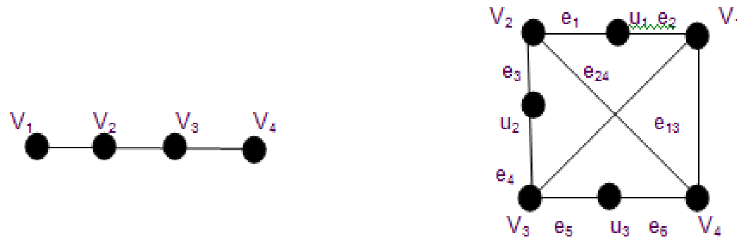

Figure 3. Path graph with four vertices and its central graph

Theorem 5.5

For any cycle graph C_n, the distance- k domination number of its central graph satisfies

$$\gamma_{\leq k}\big(C(C_n)\big) = 1 = \gamma_{c\leq k}\big(C(C_n)\big)$$

Proof.

Let the cycle graph C_n have vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $E(C_n) = \{e_1, e_2, \dots, e_n\}$, where $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$ and $e_n = v_n v_1$.

Constructing the central graph $C(C_n)$ introduces a subdivision vertex u_i for each edge e_i, resulting in the vertex set

$$V\big(C(C_n)\big) = \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_n\}$$

In $C(C_n)$, each original vertex v_i is adjacent to all vertices except its immediate neighbors v_{i-1} and v_{i+1} (indices modulo n), which are at distance two from v_i. Moreover, all subdivision vertices u_i lie within distance two of v_i. Consequently, a single vertex v_i forms a distance- k dominating set of $C(C_n)$, implying

$$\gamma_{\leq k}\big(C(C_n)\big) = 1 = \gamma_{c\leq k}\big(C(C_n)\big)$$

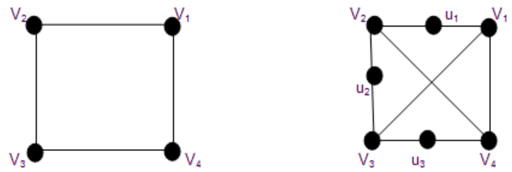

Figure 4. The star graph $K_{1,n}$ and its central graph $C\big(K_{1,n}\big)$.

Theorem 5.6

For any star graph $K_{1,n}$, the distance- k domination number of its central graph satisfies

$$\gamma_{\leq k}\big[C\big(K_{1,n}\big)\big] = 1 = \gamma_{c\leq k}\big[C\big(K_{1,n}\big)\big].$$

Proof.

Let $K_{1,n}$ have vertex set $\{v, v_1, v_2, \dots, v_n\}$, where the central vertex v has degree $\deg(v) = n$. Constructing the central graph $C\big(K_{1,n}\big)$ introduces a subdivision vertex u_i for each edge $vv_i(1 \leq i \leq n)$, yielding new edges $e_i = v_i u_i$ and $e_i' = vu_i$.

In $C\big(K_{1,n}\big)$, the central vertex v is adjacent to all subdivision vertices and lies within distance k of all other vertices. Therefore, the single vertex v constitutes a distance- k dominating set of $C\big(K_{1,n}\big)$, implying

$$\gamma_{\leq k}\big[C\big(K_{1,n}\big)\big] = 1 = \gamma_{c\leq k}\big[C\big(K_{1,n}\big)\big].$$

This holds for any $k \geq 1$.

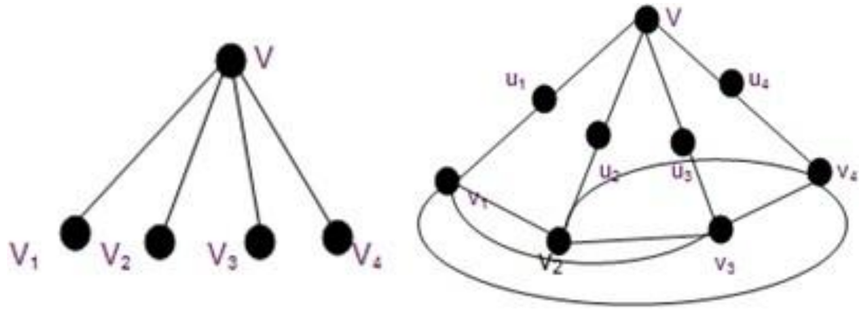

Figure 5. The star graph with $(\mathbf{1}, \boldsymbol{n})$ vertices and its corresponding middle graph.

Theorem 5.7

For any cycle graph C_n, the distance-2 domination number of its middle graph satisfies

$$\gamma_{\leq 2}\big(M(C_n)\big) = 1 = \gamma_{c\leq 2}\big(M(C_n)\big).$$

Proof.

Let $V(C_n) = \{v_1, v_2, v_3, \dots, v_n\}$ and $E(C_n) = \{e_1, e_2, \dots, e_n\}$, where $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$ and $e_n = v_n v_1$. By definition, the middle graph $M(C_n)$ has vertex set $V(C_n) \cup E(C_n)$. Each edge vertex e_i is adjacent to e_{i+1} for $1 \leq i \leq n-1$, and e_n is adjacent to e_1. Consequently, the sequence

$$v_1, e_1, v_2, e_2, v_3, e_3, \dots, v_n, e_n$$

induces a cycle of length $2n$ in $M(C_n)$.

Selecting any vertex v_i ensures that all remaining vertices of $M(C_n)$ lie within distance two from v_i. Thus,

$$\gamma_{\leq 2}\big(M(C_n)\big) = 1 = \gamma_{c\leq 2}\big(M(C_n)\big)$$

Theorem 5.8

For any star graph $K_{1,n}$, the distance- k domination number of its middle graph, for $k = 2$, is given by

$$\gamma_{\leq 2}\left(M(K_{1,n})\right) = 1 = \gamma_{c\leq 2}\left(M(K_{1,n})\right).$$

Proof.

Let $V(K_{1,n}) = \{v, v_1, v_2, \dots, v_n\}$, where v is the central vertex, and let $E(K_{1,n}) = \{e_1, e_2, \dots, e_n\}$. In the middle graph $M(K_{1,n})$, the vertex set can be expressed as

$$V\left(M(K_{1,n})\right) = \{v\} \cup \{e_i \mid 1 \leq i \leq n\} \cup \{v_i \mid 1 \leq i \leq n\}$$

Here, the vertices $\{e_1, e_2, \dots, e_n, v\}$ induce a clique of order $n+1$, while the vertices $\{v_1, v_2, \dots, v_n\}$ form an independent set. Moreover, each leaf vertex v_i is at distance two from the central vertex v, since v_i is adjacent to e_i and e_i is adjacent to v.

Hence, the singleton set $\{v\}$ constitutes a distance- 2 dominating set of $M(K_{1,n})$, establishing that

$$\gamma_{\leq 2}\left(M(K_{1,n})\right) = 1 = \gamma_{c\leq 2}\left(M(K_{1,n})\right).$$

Therefore, in general,

$$\gamma_{\leq k}\left(M(K_{1,n})\right) = 1 = \gamma_{c\leq k}\left(M(K_{1,n})\right).$$

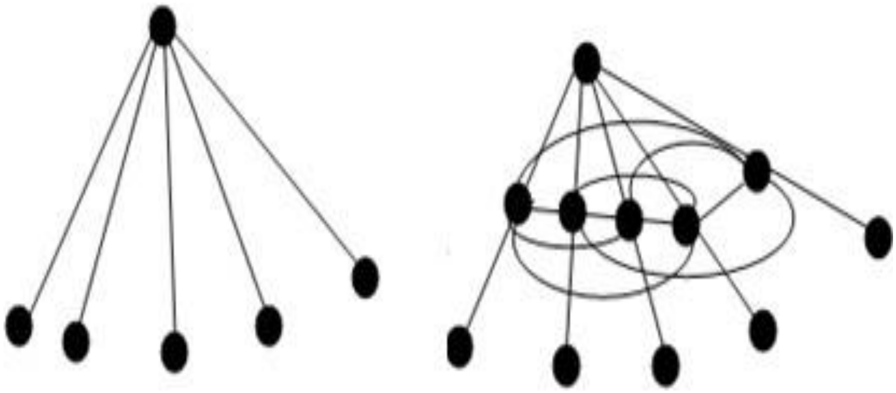

Figure 6. The star graph $K_{1,5}$ and its corresponding middle graph

Kaspar et al. [4] introduced the notion of the connected domination transition number of a graph G, defined as the difference between its connected domination number and domination number, denoted by $\tau_c(G)$.

Definition 5.9

The definition of the connected distance-domination transition number of a graph G is

$$\tau_{c\leq k}(G) = \gamma_{c\leq k}(G) - \gamma_{\leq k}(G)$$

where $\gamma_{\leq k}(G)$ and $\gamma_{c\leq k}(G)$ denote the distance- k domination number and connected distance- k domination number of G, respectively.

For example, in Example 1.2, we obtain $\tau_{c\leq 2}(G) = 0$.

6. Exact values of $\tau_{c\leq k}(G)$ for some standard graphs

The following observations summarize the connected distance- k domination transition number for several well-known graph classes:

6.1 Observation

1. For $k = 2,$ and $n \geq 3, \tau_{c\leq 2}(P_n) = \tau_{c\leq 2}(C_n)$
2. $\tau_{c\leq k}(K_n) = 0$
3. $\tau_{c\leq k}(H_n) = 0$
4. $\tau_{c\leq k}\left(K_{1,m}\right) = 0$, and $\tau_{c\leq 2}\left(K_{1,m} + G\right) = 0$ for each graph G
5. $\tau_{c\leq k}\left(K_{n,m}\right) = 0$
6. $\tau_{c\leq k}(W_n) = 0$
7. $\tau_{c\leq k}(F_n) = 0$
8. $\tau_{c\leq k}(B_n) = 0$

6.2 Proposition

For each connected graph G,

$$\tau_{c\leq k}(G) \leq \tau_c(G)$$

Proof.

Since every connected domination transition number of a graph is a particular case of the connected distance- k domination transition number, it follows directly that

$$\gamma_{\leq k}(G) \leq \gamma_{c\leq k}(G)$$

Hence, $\tau_{c\leq k}(G) \leq \tau_c(G)$.

6.3 Proposition

For every connected graph G, when $k = 2$,

$$\tau_{c\leq 2}(G) \leq \frac{|\Delta(G) - n_{\Delta}(G)|}{1 + \Delta(G)}$$

Proof.
By [4], it is known that

$$\tau_{c}(G) \leq \frac{|\Delta(G) - n_{\Delta}(G)|}{1 + \Delta(G)}$$

Applying Proposition 7.1, since $\tau_{c\leq k}(G) \leq \tau_{c}(G)$, the inequality extends to

$$\tau_{c\leq 2}(G) \leq \frac{|\Delta(G) - n_{\Delta}(G)|}{1 + \Delta(G)}$$

6.4 Proposition

For each connected graph G, when $k = 2$,

$$\tau_{c\leq 2}(G) \leq n - 3.$$

Proof.
From [4], it is established that $\tau_{c}(G) \leq n - 3$. Using Proposition 7.1,

$$\tau_{c\leq 2}(G) \leq \tau_{c}(G) \leq n - 3$$

6.5 Proposition

For each connected graph G,

$$\tau_{c\leq k}(G) \leq n - \gamma(G) - 2.$$

Proof.
By [4], the connected domination transition number satisfies

$$\tau_{c}(G) \leq n - \gamma(G) - 2.$$

Since $\tau_{c\leq k}(G) \leq \tau_{c}(G)$, the result follows.

7.6 Proposition

If G and H are connected and connected spanning subgraph of G, then

$$\tau_{c\leq k}(G) \leq \tau_{c\leq k}(H)$$

7. MINIMUM CONNECTED DISTANCE-k APPLICATIONS

7.1 Routing for School Buses

These days, practically every school has school buses to transport students to and from classes. Three key points should be highlighted among many others: (i) the bus's operating time between the school and its destination; (ii) the bus's maximum capacity at any given time; and (iii) the farthest a kid must go to board a school bus. Examine the city street map in Figure 3, where each edge corresponds to a city block. In this scenario, the school's management committee decides that no student should have to travel more than two blocks to catch a school bus, and the school is situated at the vertex beginning point. Create a school bus route that departs from the school, travels within two blocks of each student that rides it, and then returns to the school (Figure 7). This bus route obviously constitutes a minimal connected distance of two dominating sets.

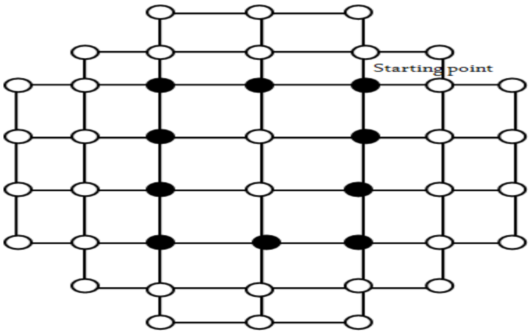

Figure 7. Bus Routing

7.2 Mobile Ad-Hoc Network

Since 1975, the linked dominating set has been a well-known topic in graph theory research. It was discovered to have significant uses in the 1990s as a virtual backbone in communication networks, particularly wireless networks. The connected dominant set has recently gained popularity as a computer science research topic.

Two dominating sets for connected distance are highly helpful for calculating routing for mobile ad hoc networks. The backbone of communications in this application is a minimum connected distance - 2 dominating set, and nodes beyond this set communicate by sending messages to neighbours inside the set.

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