International Journal of Computer Information Systems and Industrial Management Applications
ISSN 2150-7988 Volume 18(10s) 2026 1066-1074.

https://doi.org/10.70917/ijcisim-2026-xxxx

Article

On Various Path Independent Axioms of Fuzzy Choice Functions

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Abstract: In fuzzy choice theory, the study of various axioms and their interrelationships play a crucial role in the rationalization of fuzzy choice functions. In this paper we introduce union and intersection-based path independence axioms U1, U2, and U3 as well as I1, I2, and I3. and examines the connections among them. It also investigates the relationships with other important properties, such as the fuzzy congruence axiom, the fuzzy superset property and the fuzzy outcasting property.

Keywords: Fuzzy choice function; Path Independent axiom; fuzzy outcasting property; fuzzy congruence axiom.

1. INTRODUCTION

The concept of revealed preference theory was introduced by Samuelson [17] to analyse the rational behavior of individual consumer. This theory was later studied in an abstract framework by Uzawa[21] and Arrow [1] through the introduction of formal choice functions. Their work was further developed by Sen [18] and several other researchers, who considered choice functions defined on all finite non-empty subsets of a set of alternatives. Also, by considering different types of domains like finite intersection preserving etc. In this framework, the rationality of choice functions was examined using revealed preference axioms, consistency conditions and congruence axioms.

For decision making environment, Orlovsky [14] initiated the study of fuzzy preference relations, which was subsequently extended by Barrett [3] and others. Later, Banerjee [2] focused on fuzzifying only the range of choice functions and proposed three independent fuzzy congruence conditions to characterize their rationality. However, Wang demonstrated that these conditions are not mutually independent by establishing relationships among them. Further contributions tofuzzy choice theory were made by Georgescu [9] and Mordeson[13] by excluding Banerjee's approach and collaborators examined fuzzy choice functions from a different perspective. It has been shown that fuzzy choice functions satisfying certain conditions are rationalizable, and they also satisfy the Fuzzy Arrow Axiom (FAA) and the Weak Axiom of Revealed Preference (WARP)etc. In [9], Georgescu studied fuzzy choice functions in a more generalized framework, where both the domain and co-domain of the choice functions consist of fuzzy sets of alternatives. She introduced fuzzy versions of revealed preference axioms, consistency conditions and congruence axioms and utilized them to examine the rationality of fuzzy choice functions. Furthermore, she extended earlier results of Richter [16] and Suzumura [19,20] by introducing concepts such as M-rationality, G-rationality, M-normality, G-normality, and full rationality of fuzzy choice functions. The entire framework developed by Georgescu [9] is based on the following two hypotheses: (H1) for every $A \in \mathscr{B}$, $\psi(A)$ is a normal fuzzy subset of Γ; and (H2) the domain $\mathscr{B}$ contains all finite fuzzy sets of Γ. We also use these hypotheses in the present study. Path independence and other properties of rationality (in the crisp version of choice theory) have been discussed by Plott in [15] and by Lahirii in [12] . In this study we introduce various forms of path independent choice and discussed the interrelation among the different path independent axioms U_i, I_i, i= 1, 2, 3, and fuzzy Chirnoff axiom (FCA), fuzzy outcasting property and fuzzy superset property in the fuzzy version.

2. Preliminaries

In the present section, we revisit some fundamental concepts related to the residuated structure on the unit interval $I = [0, 1]$ along with essential notions of fuzzy relations. Standard references forthese topics include [4,5,11]. For any $\sigma, \omega \in I$, we denote

$\sigma \vee \omega = \max(\sigma, \omega)$ and $\sigma \wedge \omega = \min(\sigma, \omega)$. More generally, for a family $\{\sigma_i\}_{i\in I}$, we denote $\bigvee_{\{i\in I\}} \sigma_i = \sup\{\sigma_i: i\in I\}$ and $\bigwedge_{\{i\in I\}} \sigma i = \inf\{\sigma_i: i\in I\}$. Let $* : [0, 1] \times [0, 1] \to [0, 1]$ be a continuous t-norm. A binary operation $\to$ on $[0, 1]$ defined by

$$a \to b = \sup\{\gamma_1 \in [0, 1] \mid a * \gamma_1 \leq b\}$$

is referred to as the residuum (also known as an R-fuzzy implication) corresponding to the t-norm$*$.

In the following lemma we summarize some important properties of the residuation operator $\to$.

Lemma 1. [9,11] *If $\to$ is a fuzzy implication associated with a continuous t-norm $*$on I, then*

(i) $\sigma * (\sigma \to \omega) \leq \sigma \wedge \omega$

(ii) $\sigma * \omega \leq \tau \Longleftrightarrow \sigma \leq \omega \to \tau$

(iii) $\sigma \leq \omega \Longleftrightarrow \sigma \to \omega = 1$

(iv) $1 \to \sigma = \sigma$

(v) $\sigma \to 1 = 1$

(vi) $\sigma \leq \omega \Rightarrow \omega \to \tau \leq \sigma \to \tau$ *and* $\tau \to \sigma \leq \tau \to \omega$;

(vii) $(\sigma \to \omega) * (\omega \to \tau) \leq \sigma \to \tau$.

We now recall some standard terminologies related to fuzzy sets and fuzzy choice functions [9]. Let Γ be a non-empty set of alternatives. A fuzzy subset A of X is a mapping $A: \Gamma \to I$, where for each $\gamma_1 \in \Gamma$, the value $A(\gamma_1)$ represents the membership degree of γ_1 in A. Denote by $P(\Gamma)$ the collection of all crisp subsets of Γ and by $F(\Gamma)$ the family of all fuzzy subsets of Γ. Clearly, $P(\Gamma) \subseteq F(\Gamma)$.

Let B in the set of all non-zero fuzzy subsets of Γ. If $A, B \in \mathscr{B}$ be such that $A \cap B \models \phi$ and $A \cap B \in \mathscr{B}$, then B we called as the intersection preserving class of fuzzy subsets. A fuzzy set A is said to be *normal* if there exists $\gamma_1 \in \Gamma$ such that $A(\gamma_1) = 1$, and it is Called non-zero if there exists $\gamma_1 \in \Gamma$ with $A(\gamma_1) > 0$. For any $A, B \in B$, define $I(A, B) = \bigwedge_{\gamma 3 \in \Gamma} [A(\gamma 3) \to B(\gamma 3)]$, $E(A, B) = \bigwedge_{\gamma 3 \in \Gamma} [A(\gamma 3) \leftrightarrow B(\gamma 3)]$. Here, $I(A, B)$ represent the degree to which A is a subset of B, while $E(A, B)$ measures the degree of equality between A and B. It follows that $A \subseteq B$ if and only if $I(A, B) = 1$, and $A = B$ if and only if $E(A, B) = 1$. Moreover, for every $\gamma_1 \in \Gamma$, we have $I(A, B) \leq A(\gamma_1) \to B(\gamma_1)$ and $E(A, B) \leq A(\gamma_1) \leftrightarrow B(\gamma_1)$.

Definition 1. [9] A *fuzzy choice function* $\psi: \mathscr{B} \to \mathscr{B}$ is a function such that for any $A \in \mathscr{B}$, $\psi(A) \subseteq A$.

3. Axioms of Path Independent Choice

In classical choice theory, choice from a set is independent of choices that are made in one step or through intermediate sequential steps; this is treated as path independent choice. In fuzzy choice theory, only one form namely (U_1) is studied in literature [5],[6],[9] as a path independent choice function. Here, we introduce other forms (namely U_2, U_3, I_1, I_2, I_3) of path independent in the fuzzy version and find an interrelation among themselves and with their axioms of rationality of fuzzy choice function.

Definition 2. Let $\psi: \mathscr{B} \to \mathscr{B}$ be a fuzzy choice function. Then ψ satisfies

(i) U_1 axiom (Path Independent Axiom), if $\psi(A \cup B) = \psi(\psi(A) \cup \psi(B))$, for any $A, B \in \mathscr{B}$. [6,9].

(ii) U_2 axiom, if $\psi(A \cup B) = \psi(\psi(A) \cup B)$, for any $A, B \in \mathscr{B}$.

(iii) U_3 axiom, if $\psi(A \cup B) \subseteq \psi(A) \cup \psi(B)$, for any $A, B \in \mathscr{B}$.

Here, we will define other forms of path independent choice as well:

Definition 3. Let $\psi: \mathscr{B} \to \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain $\mathscr{B}$. Then ψ satisfies the

(i) I_1 axiom, if for any $A, B \in \mathscr{B}$, $\psi(A \cap B) = \psi(\psi(A) \cap \psi(B))$.

(ii) I_2 axiom, if for any $A, B \in \mathscr{B}$, $\psi(A \cap B) = \psi(\psi(A) \cap B)$.

(iii) I_3 axiom, if for any $A, B \in \mathscr{B}$, $\psi(A \cap B) \subseteq \psi(A) \cap \psi(B)$.

Lemma 2. *If a fuzzy choice function $\psi: \mathscr{B} \to \mathscr{B}$ satisfies the axiom U_1, then $\psi(\psi(A)) = \psi(A)$, for* all A, $\in \mathscr{B}$.

i.e. ψ is idempotent.

Proof. Suppose ψ satisfies axiom U_1. i.e. for any $A, B \in \mathscr{B}$, we have $\psi(A \cup B) = \psi(\psi(A) \cup \psi(B))$. By putting $A = B$, we get $\psi(A \cup A) = \psi(\psi(A) \cup \psi(A))$. This proves $\psi(A) = \psi(\psi(A))$.

Lemma 3. *let ψ fuzzy choice function ψ: $\mathscr{B} \rightarrow \mathscr{B}$ defined on an intersection preserving domain B. If C satisfies the I_1axiom, then ψ is idempotent.*

Proof. Let $A \in \mathscr{B}$. Then by I_1 axiom we have $\psi(A \cap A) = \psi(\psi(A) \cap \psi(A))$.i.e.$\psi(A) = \psi(\psi(A))$.

Theorem 1. *A fuzzy choice function ψ: $\mathscr{B} \rightarrow \mathscr{B}$ satisfies the U_1axiom if and only if it satisfies the axiom U_2.*

Proof. Suppose ψ satisfies U_1 axiom.

Let $A, B \in \mathscr{B}$. Then

$$\psi(\psi(A) \cup B) = \psi(\psi(\psi(A)) \cup \psi(B))$$

$$= \psi(\psi(A) \cup \psi(B)), \text{ by Lemma 3.}$$

$$= \psi(\psi(A) \cup \psi(B))$$

$$= \psi(A \cup B), \text{ by axiom } U_1.$$

Thus, ψ satisfies U_2.

Conversely, suppose ψ satisfies U_2. For any $A, B \in \mathscr{B}$, we have

$$\psi(A \cup B) = \psi(\psi(A) \cup B)$$

$$= \psi(B \cup \psi(A))$$

$$= \psi(\psi(B) \cup \psi(A)), \text{ by axiom } U_2.$$

Theorem 2. *Let ψ: $\mathscr{B} \rightarrow \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain $\mathscr{B}$. Then ψ satisfies the axiom I_1if and only if it satisfies axiom I_2.*

Proof. Let $A, B \in \mathscr{B}$ with $A, B \models \phi$ and $\psi(A) \cap B \models \phi$.

Then

$$\psi(\psi(A) \cap B) = \psi(\psi(\psi(A) \cap \psi(B))), \text{ by the axiom } I_1.$$

$$= \psi(\psi(A) \cap \psi(B)), \text{ by Lemma 3}$$

$$= \psi(A \cap B), \text{ by the axiom } I_1.$$

Conversely, let $A, B \in \mathscr{B}$ with $A, B \models \phi$ and $\psi(A) \cap B \models \phi$.

Then

$$\psi(A \cap B) = \psi(\psi(A) \cap B)$$

$$= \psi(B \cap \psi(A))$$

$$= \psi(\psi(B) \cap \psi(A)), \text{ by the axiom } I_2.$$

Thus, ψ satisfies the axiom I_1.

Theorem 3. *If a fuzzy choice function ψ: $\mathscr{B} \rightarrow \mathscr{B}$ satisfies axiom U_1, then it satisfies axiom U_3.*

Proof. Suppose ψ satisfies U_1. For any $A, B \in \mathscr{B}$ we have

$$\psi(A \cup B) = \psi(\psi(A)) \cup \psi(B)), \text{ by the axiom } U_1,$$

$$\subseteq \psi(A) \cup \psi(B), \text{ by FCF.}$$

Thus, ψ satisfies U_3.

□

Corollary 1. *If a fuzzy choice function ψ: $\mathscr{B} \rightarrow \mathscr{B}$ satisfies the axiom U_2, then it satisfies the axiom U_3.*

Proof. Using Theorem 1, we have, U_2 implies U_1. Therefore, again using Theorem3, we get U_2implies U_3. □

Theorem 4. *Let ψ: $\mathscr{B} \rightarrow \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain $\mathscr{B}$. If ψ satisfies the axiom I_1, then it satisfies the axiom I_3.*

Proof. Suppose ψ satisfies I_1. For any $A, B \in \mathscr{B}$ we have

$$\psi(A \cap B) = \psi(\psi(A)) \cap \psi(B)), \text{ using } I_1,$$

$$\subseteq \psi(A) \cap \psi(B), \text{ using definition of choice function.}$$

Thus, ψ satisfies I_3.

Corollary 2. *Let ψ: $\mathscr{B} \to \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain $\mathscr{B}$. If ψ satisfies the axiom I_2, then it satisfies axiom I_3.*

Proof. Using Theorem 2, we have, I_2 implies I_1. Therefore, again using Theorem 4, we get I_2implies I_3. □

Theorem 5. [5] *A fuzzy choice function ψ: $\mathscr{B} \to \mathscr{B}$ is Q-rational if and only if ψ satisfies the fuzzy path independent property and the fuzzy Condorcet property.*

Corollary 3. *A fuzzy choice function ψ: $\mathscr{B} \to \mathscr{B}$ is Q-rational if and only if ψ satisfies the the axiom $U_2$and the fuzzy Condorcet property.*

Proof. This follows from the Theorem 1.

Corollary 4. *If a fuzzy choice function ψ: $\mathscr{B} \to \mathscr{B}$ is Q-rational, then ψ satisfies the axiom$U_3$and the fuzzy Condorcet property.*

Proof. This follows from the Theorem 3.

The following examples show that the some of interrelations of the axioms U_1, U_2, U_3 and I_1, I_2, I_3do not hold in general.

Example 1. The axiom U_3 does not imply the axiom U_2.

Solution. Consider the following Example: Let $\Gamma = \{\sigma, \omega, \tau\}$, $\psi\{1/\sigma\} = 1/\sigma$, $\forall\sigma \in\Gamma$ *and* ψ $\{1/\sigma + 1/\omega\} = 1/\sigma + 0.2/\omega$, ψ $\{1/\sigma + 1/\tau\} = 1/\sigma + 1/\tau$, ψ $\{1/\omega + 1/\tau\} = 1/\omega$, ψ $(\Gamma) = 1/\sigma$, ψ $\{1/\sigma + 0.2/\omega + 1/\tau\} = 1/\sigma + 1/\omega$. This satisfies U_3. But for $A = \{1/\sigma + 1/\omega\}$ and $B = \{1/\sigma + 1/\tau\}$, then ψ $(A \cup B) = 1/\sigma \neq 1/\sigma + 1/\omega = \psi(\psi(A) \cup B)$.

Example 2. The axiom U_3 does not imply the axiom U_1.

Solution. Let $\Gamma = \{\sigma, \omega, \tau\}$, $\psi\{1/\sigma\} = 1/\sigma$, $\forall\sigma \in\Gamma$ *and* ψ $\{1/\sigma +1/\omega\} = 1/\sigma +0.4/\omega$, ψ $\{1/\sigma +1/\tau\} = 01/\sigma+1/\tau$, ψ $\{1/\omega +1/\tau\} = 1/\omega$, ψ $(\Gamma) = 1/\sigma$, ψ $\{1/\sigma+0.4/\omega +1/\tau\} = 01/\sigma+1/\omega$.This satisfies U_3. But for $A = \{1/\sigma + 1/\omega\}$ and $B = \{1/\sigma + 1/\tau\}$, then $\psi(A \cup B) = 1/\sigma \neq 1/\sigma + 1/\omega = \psi(\psi(A) \cup B)$. By Theorem 1,$U_1$ does not hold.

Example 3. Let ψ: $\mathscr{B} \to \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain $\mathscr{B}$. Then axiom I_3 does not implies axiom I_2.

Solution. Let $\Gamma = \{\sigma, \omega, \tau\}$ and $\psi\{1/\sigma\} = 1/\sigma$, $\forall\sigma \in\Gamma$ and $\psi\{1/\sigma+1/\omega\} = 1/\sigma+0.6/\omega$, ψ $\{1/\sigma+ 1/\tau\} = 1/\sigma + 1/\tau$, ψ $\{1/\omega + 1/\tau\} = 1/\omega + 1/\tau$, ψ $(\Gamma) = 1/\sigma + 0.6/\omega + 1/\tau$, ψ $\{1/\sigma + 0.2/\omega + 1/\tau \} = 1/\sigma + 1/\omega$. This satisfies I_3. But for $A = \{1/\sigma + 1/\omega\}$ and $B = \{1/\omega + 1/\tau\}$, then ψ $(A \cap B) = 1/\omega \neq 0.6/\omega = \psi(\psi(A) \cap B)$. Hence, ψ does not satisfies I_2.

Example 4. Let ψ: $\mathscr{B} \to \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain $\mathscr{B}$. Then axiom I_3 does not implies axiom I_1.

Solution. Consider the fuzzy choice functions given in Example 3, satisfies axiom I_3. But for $A = \{1/\sigma + 1/\omega\}$ and $B = \{1/\omega + 1/\tau\}$, then ψ $(A \cap B) = 1/\omega \neq 0.6/\omega = \psi(\psi(A) \cap \psi(B))$. Hence, ψ does not satisfies I_1.

Definition 4. [8] Let ψ: $\mathscr{B} \to \mathscr{B}$ be a Fuzzy choice function. Then ψ satisfies Fuzzy Chirnoff Axiom (FCA), if for any $A, B \in\mathscr{B}$ and $\gamma \in\Gamma$, we have

$$I(A, B) \wedge A(\gamma) \wedge\psi(B)(\gamma) \leq I(\psi(B) \cap A, \psi(A)).$$

Theorem 6. *If a fuzzy choice function ψ: $\mathscr{B} \to \mathscr{B}$ satisfies FCA, then it satisfies the axiom U_3.*

Proof. Consider for any $\gamma \in\Gamma$,

$$\psi (A \cup B) (\gamma) = \psi (A \cup B) (\gamma) \wedge (A \cup B) (\gamma),$$

$$\leq \psi (A \cup B) (\gamma) \wedge (A(\gamma) \vee B(\gamma))$$

$$\leq [\psi (A \cup B) (\gamma) \wedge A(\gamma)] \vee [\psi (A \cup B) (\gamma) \wedge B(\gamma))] \quad (1)$$

We have, $A \subseteq A \cup B$ and $B \subseteq A \cup B$, Therefore, $I (A, A \cup B) = 1$ and $I (B, A \cup B) = 1$

$$\psi (A \cup B) (\gamma) \wedge A(\gamma) = \psi (A \cup B) (\gamma) \wedge A(\gamma) \wedge I (A, A \cup B)$$

$$\leq \psi (A \cup B) (\gamma) \wedge A(\gamma) \leq I (A \cap \psi (A \cup B), \psi(A)) \qquad \text{by FCA}$$

$$\leq \psi (A \cup B) (\gamma) \wedge A(\gamma) \leq A(\gamma) \wedge\psi (A \cup B) (\gamma) \to \psi(A)(\gamma).$$

By property of Lemma 1(ii), this implies that,

$$\psi (A \cup B) (\gamma) \wedge A(\gamma) \wedge A(\gamma) \wedge\psi (A \cup B) (\gamma) \leq \psi(A)(\gamma).$$

This gives us,

$\psi\ (A \cup B)\ (\gamma) \wedge A(\gamma) \leq \psi(A)(\gamma)$. (2)

Using the same approach,

$\psi\ (A \cup B)\ (\gamma) \wedge B(\gamma) \leq \psi(B)(\gamma)$ (3)

Upon substitution of these values into equation (1), we arrive at

$\psi\ (A \cup B)\ (\gamma) \leq \psi(A)(\gamma) \vee \psi(B)(\gamma) = (\psi(A) \cup \psi(B))\ (\gamma)$.

Hence,

$\psi\ (A \cup B)\ (\gamma) \leq (\psi(A) \cup \psi(B))\ (\gamma)$.

Above inequality is true for all $\gamma \in \Gamma$, therefore,

$\psi\ (A \cup B) \subseteq \psi(A) \cup \psi(B)$,

i.e. U_3.

Theorem 7. *Let ψ: $\mathscr{B} \to \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain $\mathscr{B}$. If ψ satisfies FCA, then it satisfies axiom I_3.*

Proof. Consider for any $\gamma \in \Gamma$,

$$\begin{aligned}\psi\ (A \cap B)\ (\gamma) &= \psi\ (A \cap B)\ (\gamma) \wedge (A \cap B)\ (\gamma),\\ &\leq \psi\ (A \cap B)\ (\gamma) \wedge (A(\gamma) \vee B(\gamma))\\ &\leq [\psi\ (A \cap B)\ (\gamma) \wedge A(\gamma)] \vee [\psi\ (A \cap B)\ (\gamma) \wedge B(\gamma))] \qquad (4)\end{aligned}$$

We have, $A \subseteq A \cap B$ and $B \subseteq A \cap B$, Therefore, $I\ (A, A \cap B) = 1$ and $I\ (B, A \cap B) = 1$

$$\begin{aligned}\psi\ (A \cap B)\ (\gamma) \wedge A(\gamma) &= \psi\ (A \cap B)\ (\gamma) \wedge A(\gamma) \wedge I\ (A, A \cap B)\\ &\leq \psi\ (A \cap B)\ (\gamma) \wedge A(\gamma) \leq I\ (A \cap \psi\ (A \cap B), \psi(A)), \text{ by FCA}\\ &\leq \psi\ (A \cap B)\ (\gamma) \wedge A(\gamma) \leq A(\gamma) \wedge \psi\ (A \cap B)\ (\gamma) \to \psi(A)(\gamma)\ .\end{aligned}$$

By property of Lemma 1(ii), this implies that,

$\psi\ (A \cap B)\ (\gamma) \wedge A(\gamma) \wedge A(\gamma) \wedge \psi\ (A \cap B)\ (\gamma) \leq \psi(A)(\gamma)$.

This gives us,

$\psi\ (A \cap B)\ (\gamma) \wedge A(\gamma) \leq \psi(A)(\gamma)$. (5)

With a similar argument, we will get

$\psi\ (A \cap B)\ (\gamma) \wedge B(\gamma) \leq \psi(B)(\gamma)$ (6)

Inserting these results into equation (1) yields,

$\psi\ (A \cap B)\ (\gamma) \leq \psi(A)(\gamma) \vee \psi(B)(\gamma) = (\psi(A) \cap \psi(B))\ (\gamma)$.

Hence,

$\psi\ (A \cap B)\ (\gamma) \leq (\psi(A) \cap \psi(B))\ (\gamma)$.

Thus, ψ satisfies the axiom I_3. □

The following examples shows that the converse parts of Theorems 6 and 7 do not hold.

Example 5. The axiom U_3 does not imply FCA.

Solution. Consider the fuzzy choice function defined in Example 6. Then ψ satisfies axiom U_3 but not FCA as for $\gamma = \omega$, $A = \{1/\omega + 1/\tau\}$, $B = \Gamma$, $I(A, B) \wedge A(\omega) \wedge \psi(B)(\omega) = 1 \not\models 0 = I(\psi(B) \cap A, \psi(A))$.

The following example shows that axiom U_1 and FCA are independent.

Example 6. Axiom U_1 does not imply FCA.

Solution. Let $\Gamma = \{\sigma, \omega, \tau\}$. *Define* the choice function ψ as follows $\psi(\{\sigma\}) = \{1/\sigma\}$ $\forall \sigma \in \Gamma$, $\psi\ (\Gamma) = \{1/\sigma + 1/\omega\}$, $\psi(\{\sigma, \omega\}) = \{1/\sigma + 1/\omega\}$, $\psi(\{\sigma, \tau\}) = \{1/\sigma + 1/\tau\}$, $\psi(\{\omega, \tau\}) = \{1/\omega\}$.Then ψ satisfies the axiom U_1 but not FCA as for $\gamma = \omega$, $A = \{1/\omega + 1/\tau\}$, $B = \Gamma$, $I(A, B) \wedge A(\omega) \wedge \psi(B)(\omega) = 1 \not\models 0 = I(\psi(B) \cap A, \psi(A))$.

Example 7. FCA does not imply axiom U_1.

Solution. Let $\Gamma = \{\sigma, \omega, \tau\}$, $\psi(\Gamma) = \{1/\sigma\}$ and $\psi(A) = A$ otherwise. Here ψ satisfies FCA but not U_1 as for $A = \{1/\sigma + 1/\omega\}$, $B = \Gamma$,$\psi(A \cup B) = \psi(\Gamma) = 1/\sigma + 1/\omega \not\models \{1/\sigma + 1/\omega\} = \psi(\psi(A) \cup \psi(B))$.

Example 8. Let $\psi: \mathscr{B} \to \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain $\mathscr{B}$. Then axiom I_3 does not imply FCA.

Solution. Consider the fuzzy choice function given in Example 3, This satisfies the axiom I_3. But for $A = \{1/\sigma + 1/\omega\}$, $B = \Gamma$ and for $\gamma = \sigma$, $I(A, B) \wedge A(\sigma) \wedge \psi(B)(\sigma) = 1 > I(\psi(B) \cap A, \psi(A)) = 0.2$. Hence, ψ does not satisfies FCA.

Example 9. Let $\psi: \mathscr{B} \to \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain B. Then the axiom I_1 does not imply FCA.

Solution. Let $\Gamma = \{\sigma, \omega, \tau\}$. Define the choice function ψ as follows: $\psi(\{\sigma\}) = \{1/\sigma\}$ $\forall \sigma \in \Gamma$, $\psi(\Gamma) = \{1/\sigma + 1/\omega + 1/\tau\}$, $\psi(\{\sigma, \omega\}) = \{1/\sigma + 1/\omega\}$, $\psi(\{\sigma, \tau\}) = \{1/\sigma\}$, $\psi(\{\omega, \tau\}) = \{1/\omega + 1/\tau\}$. Then ψ satisfies Property I_1 but not FCA as for $\gamma = \omega$, $A = \{1/\omega + 1/\tau\}$, $B = \Gamma$, $I(A, B) \wedge A(\omega) \wedge \psi(B)(\omega) = 1 \not\models 0 = I(\psi(B) \cap A, \psi(A))$.

Example 10. Let $\psi: \mathscr{B} \to \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain B. Then FCA does not imply the axiom I_1.

Solution. Consider the fuzzy choice function given in Example 7, satisfies FCA. But for $A = \{1/\sigma + 1/\omega\}$ and $B = \Gamma$, then $\psi(A \cap B) = \{1/\sigma + 1/\omega\} \not\models 1/\sigma = \psi(\psi(A) \cap \psi(B))$. Hence, ψ does not satisfies I_1.

Example 11. Let $\psi: \mathscr{B} \to \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain B. Then I_1 does not imply FCA.

Solution. Let $\Gamma = \{\sigma, \omega, \tau\}$. Define the choice function ψ as follows $\psi(\{\sigma\}) = \{1/\sigma\}$ $\forall \sigma \in \Gamma$, $\psi(\Gamma) = \{1/\sigma\}$, $\psi(\{\sigma, \omega\}) = \{1/\sigma\}$, $\psi(\{\sigma, \tau\}) = \{1/\sigma\}$, $\psi(\{\omega, \tau\}) = \{1/\omega\}$. Then ψ satisfies Property I_1 but not FCA as for $\gamma = \sigma$, $A = \{1/\omega + 1/\tau\}$, $B = \Gamma$, $I(A, B) \wedge A(\sigma) \wedge \psi(B)(\sigma) = 1 \not\models 0 = I(\psi(B) \cap A, \psi(A))$.

4. Fuzzy Outcasting and Superset Properties

In this section, we introduce fuzzy choice functions and find their relationship with path independent axioms.

Definition 5. Let $\psi: \mathscr{B} \to \mathscr{B}$ be a fuzzy choice function. Then ψ satisfies

(i) fuzzy outcasting property (FOP), if for any $A, B \in \mathscr{B}$, $I(\psi(B), A) \wedge I(A, B) \leq E(\psi(A), \psi(B))$.

(ii) fuzzy superset property (FSP), if for any $A, B \in \mathscr{B}$, $I((\psi(B), \psi(A)) \wedge I(\psi(A), A) \wedge I(A, B) \leq E(\psi(A), \psi(B))$.

Theorem 8. *If a fuzzy choice function $\psi: \mathscr{B} \to \mathscr{B}$ satisfies FOP, then it satisfies FSP.*

Proof. Suppose ψ satisfies FOP. For any $A, B \in \mathscr{B}$,

$I(\psi(B), \psi(A)) \wedge I(\psi(A), A) \wedge I(\psi(A), A)$

$= \bigwedge_{\gamma 3 \in \Gamma}[\psi(B)(\gamma 3) \to \psi(A)(\gamma 3)] \wedge \bigwedge_{\gamma 3 \in \Gamma}[\psi(A)(\gamma 3) \to A(\gamma 3)] \wedge \bigwedge_{\gamma 3 \in \Gamma}[A(\gamma 3) \to B(\gamma 3)]$

$= \bigwedge_{\gamma 3 \in \Gamma}\{[\psi(B)(\gamma 3) \to \psi(A)(\gamma 3)] \wedge [\psi(A)(\gamma 3) \to A(\gamma 3)] \wedge [A(\gamma 3) \to B(\gamma 3)]\}$

$\leq \bigwedge_{\gamma 3 \in \Gamma}\{[\psi(B)(\gamma 3) \to A(\gamma 3)] \wedge [A(\gamma 3) \to B(\gamma 3)]\})]\}$,by Lemma 1, (vii).

$\leq \bigwedge_{\gamma 3 \in \Gamma}[\psi(B)(\gamma 3) \to A(\gamma 3)] \wedge \bigwedge_{\gamma 3 \in \Gamma}[A(\gamma 3) \to B(\gamma 3)]$

$\leq I(\psi(B), A) \wedge I(A, B)$

$\leq E((\psi(B), \psi(A))$, by FOP.

Thus, ψ satisfies the FSP. □

The converse part of the above theorem does not hold.

Example 12. FSP does not satisfy FOP.

Solution. Consider the following Example: Let $\Gamma = \{\sigma, \omega, \tau\}$, $\psi\{1/\sigma\} = 1/\sigma$, $\forall \sigma \in \Gamma$ *and* $\psi\{1/\sigma + 1/\omega\} = 1/\sigma + 1/\omega$, $\psi\{1/\sigma + 1/\tau\} = 1/\sigma + 1/\tau$, $\psi\{1/\omega + 1/\tau\} = 1/\omega + 1/\tau$, $\psi(\Gamma) = 1/\sigma + 0.2/\omega + 0.6/\tau$, This satisfies FSP but for $A = \{1/\sigma + 1/\omega\}$ and $B = \Gamma$, FOP does not satisfy as $I(\psi(B), A) \wedge I(A, B) = 1 \geq 0.2 = E((\psi(B), \psi(A)))$.

Example 13. FSP does not satisfy the axiom U_1.

Solution. Consider the following Example: Let $\Gamma = \{\sigma, \omega, \tau\}$, $\psi\{1/\sigma\} = 1/\sigma$, $\forall \sigma \in \Gamma$ *and* $\psi\{1/\sigma + 1/\omega\} = 1/\sigma + 1/\omega$, $\psi\{1/\sigma + 1/\tau\} = 1/\sigma + 1/\tau$, $\psi\{1/\omega + 1/\tau\} = 1/\omega + 1/\tau$, $\psi\{1/\sigma + 1/\omega + 0.6/\tau\} = 1/\sigma$, $\psi(\Gamma) = 1/\sigma + 0.2/\omega + 0.6/\tau$. This satisfies FSP but for $A = \{1/\sigma + 1/\omega\}$ and $B = \Gamma$, U_1 does not satisfy as $\psi(A \cup B) = 1/\sigma + 0.2/\omega + 0.6/\tau \neq 1/\sigma = \psi(\psi(A) \cup \psi(B))$.

Example 14. Axiom U_1 does not satisfies FSP.

Solution. Let $\Gamma = \{\sigma, \omega, \tau\}$. Define the choice function ψ as $\psi(\{\sigma\}) = \{1/\sigma\} \forall \sigma \in \Gamma$, $\psi(\Gamma) = \{1/\sigma + 1/\omega + 0.2/\tau\}$, $\psi(\{\sigma, \omega\}) = \{1/\sigma + 1/\omega\}$, $\psi(\{\sigma, \tau\}) = \{1/\sigma + 1/\tau\}$, $\psi(\{\omega, \tau\}) = \{1/\omega\}$, Then ψ satisfies axiom U_1.

Example 15. FSP does not satisfies the axiom U_3.

Solution. Consider the fuzzy choice function defined in Example 17, which satisfies FSP but does not satisfy the axiom U_3 as for $A = \{1/\sigma + 1/\omega\}$ and $B = \Gamma$ *and* $\psi(A \cup B) = 1/\sigma + 0.2/\omega + 0.6/\tau \nsubseteq 1/\sigma + 1/\omega = \psi(\psi(A) \cup \psi(B))$.

Example 16. Axiom U_3 does not satisfies FSP.

Solution. Consider the fuzzy choice function defined in Example 1, which satisfies axiom U_3 but does not satisfy FSP as for $A = \{1/\sigma + 1/\omega\}$ and $B = \Gamma, I((\psi(B), \psi(A)) \wedge I(\psi(A), A) \wedge I(A, B)) = 1 > 0 = E(\psi(A), \psi(B))$.

Example 17. Let $\psi: \mathscr{B} \rightarrow \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain $\mathscr{B}$. Then FSP does not satisfies the axiom I_1.

Solution. Let $\Gamma = \{\sigma, \omega, \tau\}$, $\psi\{1/\sigma\} = 1/\sigma$, $\forall \sigma \in \mathrm{I}$ *and* $\psi\{1/\sigma + 1/\omega\} = 1/\sigma + 1/\omega$, $\psi\{1/\sigma + 0.2/\omega\} = 1/\sigma + 0.2/\omega$, $\psi\{1/\sigma + 1/\tau\} = 1/\sigma + 1/\tau$, $\psi\{1/\omega + 1/\tau\} = 1/\omega + 1/\tau$, $\psi\{1/\sigma + 1/\omega + 0.6/\tau\} = 1/\sigma$, $\psi(\Gamma) = 1/\sigma + 0.2/\omega + 0.6/\tau$. This satisfies FSP but for $A = \{1/\sigma + 1/\omega\}$ and $B = \Gamma, I_1$ does not satisfy as $\psi(A \cap B) = 1/\sigma + 1/\omega \neq 1/\sigma + 0.2/\omega = \psi(\psi(A) \cap \psi(B))$.

Example 18. Let $\psi: \mathscr{B} \rightarrow \mathscr{B}$ be a fuzzy choice function defined on an intersection preserving domain $\mathscr{B}$. Then FSP does not satisfies the axiom I_3.

Solution. Consider the fuzzy choice function given in Example 17. This satisfies FSP but for $A = \{1/\sigma + 1/\omega\}$ and $B = \Gamma$, I_3 does not satisfy as $\psi(A \cap B) = 1/\sigma + 1/\omega \nsubseteq 1/\sigma + 0.2/\omega = \psi(A) \cap \psi(B)$.

5. Conclusion

This paper introduces various path independent U1, U2, U3 and I1, I2, I3 for fuzzy choice and discusses the fundamental role of interrelationships among path independent and axioms in the rationalization of fuzzy choice functions. By analysing the connections between them discusses the nature of fuzzy congruence axiom, fuzzy superset property and fuzzy outcasting property. These findings contribute to a more unified framework for fuzzy choice theory and will offer insights that may support further theoretical developments in the field.

6. Acknowledgment

The first author sincerely acknowledges the financial assistance received from the Council of Scientific & Industrial Research (CSIR), New Delhi, in the form of a Senior Research Fellowship (SRF), which supported this research work.

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