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Article

The Intrinsic Psychological Health of Young Adults Through A Truncated Weibull Model

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Abstract: The psychological data collected through questionnaires, extreme events such as total sickness or complete health, are very rare. This leads to an almost essential restriction of the data domain and hence demands truncation of data. Also, the data is essentially qualitative in nature (usually collected using a Likert scale). The objectives of the study were to model qualitative data using a continuous truncated Weibull model and to compare the three estimation techniques, namely, the Maximum Likelihood estimation, the Method of Moments (Cran) and the Bayesian Estimation method, all used under different circumstances, in case of this model. The choice of Weibull model was based on AIC and BIC criteria and was validated through the Kolmogorov-Smirnov test, prior to truncation. The parameters of the truncated model were estimated using the three estimation techniques on the data collected through the Strength and Difficulty Questionnaire (SDQ) 17+ extended version during Covid-19 pandemic period. The difficulty score component of SDQ (range 0-40) was truncated for total difficulty scores (i) less than 2; and (ii) more than 30. The estimates obtained through the three techniques were consistent up to two places after decimal.
Keywords: Difficulty Score, Estimation Techniques, Qualitative data, Strength and Difficulty Questionnaire, Truncated Weibull Distribution.

1. INTRODUCTION

Statistical distributions are an integral part of statistical theory and modelling, which are used to describe the properties of given data or theoretically propose models over a given domain. However, not always are the complete domains available, and researchers have access to only a part of it. Sometimes the distribution domain is restricted in the sense that the event of occurrence of the extreme values is highly unlikely. In such situations, a truncated distribution is the choice. Depending upon restrictions, a distribution can either be left /right, or both sides truncated [1, 2]. Truncated distributions are frequently used to model psychological data that are qualitative in nature. The responses are scored, and the scores are categorized to identify the extent of problems. In this kind of data, extreme values do not occur in general, as no individual is perfectly healthy or totally sick.

Weibull distribution, identified by a shape and a scale parameter, is very flexible and hence can be applied in different areas such as reliability, survival analysis, economic conditions, meteorology, quality control, fatigue analysis, medical and bio-statistical data analysis [3, 4]. Baghestani et al. [5] applied a Weibull model on survival pattern of breast cancer patients.

One of the behavioural screening tools, the Strengths and Difficulties Questionnaire (SDQ), is designed to capture the perspective of children and young adults in terms of strengths and difficulties of their personality [6]. The basic version of the questionnaire consists of five scales, four of which, namely, Conduct problems, Hyperactivity-inattention, Emotional symptoms, and Peer problems, pertain to the 'Difficulty' aspect of one's personality and the fifth, namely, the Prosocial behaviour examines their social strength. Each scale consists of five items. The basic version is meant to examine the extent of social dysfunction in the respondent. The extended version, along with the social dysfunction, has another 8 items and examines the behavioural problems through the 'Impact score'.

Literature review

Dey et al. [7] studied the two-parameter Rayleigh distribution and estimated its parameters through several estimation techniques, viz, maximum likelihood estimators, method of moments, least squares estimators, Bayes estimators and L-moment estimators. They compared the estimators with respect to mean squared error and bias. Their study indicated that the Bayes estimators under non-informative priors were better than others with respect to both the mean squared error and the bias. The maximum likelihood estimators also performed satisfactorily.

Sabharwal et al. [8] proposed the same estimation techniques as used by Dey et al. [7], to estimate the parameters of the two-parameter Weibull distribution under the null and alternative hypotheses. Among the methods considered, the moment estimates obtained by the method of moments often yielded smaller standard errors, suggesting higher precision in estimation. Bayesian estimates were found to be competitive, while maximum likelihood estimates provided stable and consistent results across different scenarios.

Bihea [9] examined the performance of different estimation methods, namely, the method of moments, the method of maximum likelihood, and the method of moments by Cran for a new transmuted Rayleigh distribution. In their study, a comprehensive simulation analysis based on mean squared error demonstrated that maximum likelihood estimation tends to perform better for smaller sample sizes, whereas method of moments shows improved performance as the sample size increases.

Jahan et al. [10] demonstrated a higher efficacy of the truncated model while using a Weibull distribution (WD) and the lower-upper-truncated (double truncated) Weibull distribution (TWD) to model wind speed data in Bangladesh. Abubakar [11] applied a Bayesian approach to the Weibull distribution with a gamma prior on the shape parameter to model to estimate the expected number of claims under insurance policies

In another study by Sabharwal et al [12], the same qualitative data as used in this study was modelled using continuous Weibull and normal distribution (using power transformation). With the help of a bivariate normal model, they extracted incomplete information from one questionnaire through scores of another correlated questionnaire, even if the information was collected independently.

Present work

In the present study, discrete data (ranging from 0-40 for 20 items, each item scored on a Likert scale of range 0-2) were modelled using a continuous distribution. A double truncated Weibull distribution model was applied on a data regarding the psychological health of young adults in India during the COVID-19 pandemic period in India, collected through three independent surveys using SDQ 17+ extended version. The truncation was applied to the difficulty score component of SDQ, and the truncation criteria were:

1. Total difficulty scores less than 2, i.e. left truncation; and

2. Total difficulty scores more than 30, i.e. right truncation.

The truncation limits were chosen because the frequencies of these extreme classes were small enough to be considered to be rare. The selection of the model was based on AIC and BIC values of some commonly used distributions. The distribution selected on the lowest AIC and BIC values was then tested for adequate fit using the Kolmogorov-Smirnov test. The model parameters were estimated using three estimation techniques, namely the Maximum Likelihood estimation, the Cran's estimation and the Bayesian estimation. The probabilities for each severity band were calculated using the estimated parameters obtained through the three estimation techniques for both the complete and the truncated models. The estimated probabilities were compared with the observed ones. The truncated and complete model values were compared to the observed values. To the best of my knowledge, no previous study has compared parameter estimation techniques for the double truncated continuous model.

The paper consists of five sections. Section 1 is the introduction of the study; Section 2 is about the material and methods used in the study. Section 3 presents the results; Section 4 is the discussion. Section 5 concludes the study.

2. MATERIAL AND METHODS

2.1 Material

The data were obtained from conducting surveys on the graduate/undergraduate students in India, by administering SDQ 17+ self-reported extended version thrice during Covid-19 period. The first survey (May-June 2020) had 1020 responses, the second survey (October 2020) had 734, and the third survey (October 2021-February 2022) had 934 responses. The surveys aimed at measuring the extent of psychological difficulties faced by the young adults in the light of the unprecedented spread of disease, lockdown, quarantine and isolation, closure of schools, colleges and all social life activities [6].

The SDQ has 25 items in it, which are evaluated on a Likert scale of 0-1-2, (ranging from 'not true', to 'certainly true'). These 25 items are divided into five scales with five items in each scale, among which the 'Prosocial behaviour' measures the strengths of the respondents and the other four scales, namely, 'Emotional symptoms', 'Conduct problems', 'Hyperactivity-inattention' and the 'Peer problems' scales, are related to the 'Difficulties' aspect of the questionnaire and represent the behavioural problems. Scale scores (ranging from 0-10) are sum of the respective item scores; the total difficulty score (ranging from 0-40) is the sum of scores of four behavioural scales. The difficulty score is categorised into three severity bands: the 'Normal' band (0-15) - a normal state of mental health, the 'Borderline' band (16-19) -some problems and the 'Abnormal' band (20-40)- severe problem(s) [13].

2.2 Methods

2.2.1 Two-Parameter Weibull Distribution Functions and Moments

Let $X \sim W(\alpha, \beta)$ is a two-parameter Weibull random variable, with α as the scale and β as the shape parameter. The probability density function (PDF) $f(x)$ is

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}; \qquad 0 < x < \infty,\ \alpha > 0,\ \beta > 0 \qquad (1)$$

and the cumulative distribution function (CDF) $F(x)$ is

$$F(x) = 1 - exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\} \qquad (2)$$

The r^{th} raw moment of X is given by:

$$E(X^r) = \alpha^r\, \Gamma\left(\frac{r}{\beta} + 1\right)\ ;\ r = 1,2, \ldots \qquad (3)$$

where Γ is the Gamma function defined by:

$$\Gamma(n) = \int_0^{\infty} x^{n-1}\, e^{-x} dx \qquad (4)$$

In particular,

$$E(X) = \alpha\, \Gamma\left(\frac{1}{\beta} + 1\right) \text{ ; and} \qquad (5)$$

$$Var(X) = \alpha^2\left[\Gamma\left(\frac{2}{\beta} + 1\right) - \Gamma\left(\frac{1}{\beta} + 1\right)^2\right] \qquad (6)$$

2.2.2 Truncated Two-Parameter Weibull Distribution and its Moments

Let $X \sim LRTWD(\alpha, \beta)$ is a left-right truncated Weibull variable with parameters (α, β) defined over the interval $[u, v]$ where the truncation is over 'u' classes from the left and the classes after 'v' on the right. The probability and cumulative distribution functions of X are given by:

$$f(x|\alpha, \beta) = \frac{f(x)}{F(v) - F(u)}; \quad x > 0, u \le x \le v$$

$$= \frac{\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}}{exp\left\{-\left(\frac{u}{\alpha}\right)^{\beta}\right\} - exp\left\{-\left(\frac{v}{\alpha}\right)^{\beta}\right\}} \qquad (7)$$

and

$$F(x|\alpha,\beta) = \frac{exp\left\{-\left(\frac{u}{\alpha}\right)^{\beta}\right\} - exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}}{exp\left\{-\left(\frac{u}{\alpha}\right)^{\beta}\right\} - exp\left\{-\left(\frac{v}{\alpha}\right)^{\beta}\right\}} \tag{8}$$

For $u = 0, v = \infty$; $LRTWD(\alpha,\beta)$ reduces to a $WD(\alpha,\beta)$ distribution.

The r^{th} raw moment of the $LRTWD(\alpha,\beta)$ is:

$$E(X^r) = \frac{\alpha^r}{e^{-\left(\frac{u}{\alpha}\right)^{\beta}} - e^{-\left(\frac{v}{\alpha}\right)^{\beta}}}\left[\Gamma\left\{\frac{r}{\beta}+1, \left(\frac{v}{\alpha}\right)^{\beta}\right\} - \Gamma\left\{\frac{r}{\beta}+1, \left(\frac{u}{\alpha}\right)^{\beta}\right\}\right] \tag{9}$$

$\Gamma(s, v)$is the incomplete Gamma function.

In particular,

$$E(X) = \frac{\alpha}{e^{-\left(\frac{u}{\alpha}\right)^{\beta}} - e^{-\left(\frac{v}{\alpha}\right)^{\beta}}}\left[\Gamma\left\{\frac{1}{\beta}+1, \left(\frac{v}{\alpha}\right)^{\beta}\right\} - \Gamma\left\{\frac{1}{\beta}+1, \left(\frac{u}{\alpha}\right)^{\beta}\right\}\right] \tag{10}$$

$$Var(X) = \frac{\alpha^2}{e^{-\left(\frac{u}{\alpha}\right)^{\beta}} - e^{-\left(\frac{v}{\alpha}\right)^{\beta}}}\left[\left[\Gamma\left\{\frac{2}{\beta}+1, \left(\frac{v}{\alpha}\right)^{\beta}\right\} - \Gamma\left\{\frac{2}{\beta}+1, \left(\frac{u}{\alpha}\right)^{\beta}\right\}\right] - \left[\Gamma\left\{\frac{1}{\beta}+1, \left(\frac{v}{\alpha}\right)^{\beta}\right\} - \Gamma\left\{\frac{1}{\beta}+1, \left(\frac{u}{\alpha}\right)^{\beta}\right\}\right]^2\right] \tag{11}$$

The median m_T of a Left-Right truncated Weibull distribution is calculated using the equation:

$$m_T = \alpha\left[-\ln\left(1 - \frac{F(V)+F(U)}{2}\right)\right]^{\frac{1}{\beta}} \tag{12}$$

where,

$$F(U) = 1 - \exp\left[-\left(\frac{U}{\alpha}\right)^{\beta}\right]; \text{ and } F(V) = 1 - \exp\left[-\left(\frac{V}{\alpha}\right)^{\beta}\right].$$

2.2.3 Estimation of the Model Parameters

Using the three approaches of estimation of unknown parameters viz., the Method of Maximum Likelihood Estimation, the Bayesian Estimation and Moments Estimation by Cran, the unknown parameters of the $LRTWD(\alpha,\beta)$ have been obtained.

2.2.3.1 Maximum Likelihood Estimation Method

The model likelihood function for the *i.i.d.* $LRTWD(\alpha,\beta)$ variables $X_1, X_2, \ldots X_n$ is:

$$l(\alpha,\beta|x_1,x_2,\ldots,x_n) = \prod_{i=1}^{n} f(x_i|\alpha,\beta) \tag{13}$$

$$= \prod_{i=1}^{n} \left\{ \frac{\frac{\beta}{\alpha}\left(\frac{x_i}{\alpha}\right)^{\beta-1} exp\left(-\left(\frac{x_i}{\alpha}\right)^{\beta}\right)}{exp\left(-\left(\frac{u}{\alpha}\right)^{\beta}\right) - exp\left(-\left(\frac{v}{\alpha}\right)^{\beta}\right)} \right\} \quad (14)$$

The log-likelihood function (LLF) is:

$$L(\alpha, \beta | \underline{x}) = n\left(log\,\beta - log\,\alpha - log\left\{exp\left(-\left(\frac{u}{\alpha}\right)^{\beta}\right) - exp\left(-\left(\frac{v}{\alpha}\right)^{\beta}\right)\right\}\right) + \sum_{i=1}^{n}\left((\beta - 1)(log(x_i) - log(\alpha)) - \left(\frac{x_i}{\alpha}\right)^{\beta}\right) \quad (15)$$

Differentiating Equation (15) with respect to α and β and setting the result to zero, we have:

$$\frac{\partial L(\alpha,\beta|\underline{x})}{\partial \alpha} = 0 = -\frac{n\beta}{\alpha} + \beta \sum_{i=1}^{n}\left(\frac{x_i{}^{\beta}}{\alpha^{\beta+1}}\right) - \frac{n\beta\left\{\frac{u^{\beta}}{\alpha^{\beta+1}} exp\left(-\left(\frac{u}{\alpha}\right)^{\beta}\right) - \frac{v^{\beta}}{\alpha^{\beta+1}} exp\left(-\left(\frac{v}{\alpha}\right)^{\beta}\right)\right\}}{exp\left\{-\left(\frac{u}{\alpha}\right)^{\beta}\right\} - exp\left\{-\left(\frac{v}{\alpha}\right)^{\beta}\right\}} \quad (16)$$

$$\frac{\partial L(\alpha,\beta|\underline{x})}{\partial \beta} = 0 = \frac{n}{\beta} + \sum_{i=1}^{n}[log(x_i) - log(\alpha)] - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} log\left(\frac{x_i}{\alpha}\right)$$

$$- \frac{n\left\{\left(\frac{v}{\alpha}\right)^{\beta} log\left(\frac{v}{\alpha}\right) exp\left(-\left(\frac{v}{\alpha}\right)^{\beta}\right) - \left(\frac{u}{\alpha}\right)^{\beta} log\left(\frac{u}{\alpha}\right) exp\left(-\left(\frac{u}{\alpha}\right)^{\beta}\right)\right\}}{exp\left\{-\left(\frac{u}{\alpha}\right)^{\beta}\right\} - exp\left\{-\left(\frac{v}{\alpha}\right)^{\beta}\right\}} \quad (17)$$

The maximum likelihood estimates $\hat{\alpha}$ and $\hat{\beta}$ are then obtained by solving these equations as there is no closed-form solution for Equations (16) and (17).

2.2.3.2 Bayesian Estimation Method

Using Lindley's approximation technique, Bayes' estimates of the unknown scale and shape parameters of $LRTWD(\alpha, \beta)$ are obtained under the squared error loss function with a standard non-informative prior [14, 15]:

$$h(\alpha, \beta) \propto \left(\frac{1}{\alpha\beta}\right) \quad (18)$$

For an unknown parameter θ, to be estimated by an estimator $\widehat{\theta}$, the squared error loss function (SELF) is given by:

$$L_{SELF} = \left(\hat{\theta} - \theta\right)^2 \quad (19)$$

The Bayes estimator of θ under the squared loss function is the posterior mean given by:

$$\hat{\theta}_{BS} = E(\theta|x) \quad (20)$$

This loss function is symmetrical in nature as it gives equal weightage to both over- and underestimation.

The joint posterior distribution of (α, β) is given as:

$$p(\alpha, \beta|\underline{x}) \propto l(\alpha, \beta|\underline{x})h(\alpha, \beta) \quad (21)$$

where $l(\alpha, \beta|\underline{x})$ is the likelihood function. The joint posterior distribution of α and β given the sample is obtained as:

$$p^*(\alpha, \beta|\underline{x}) = \frac{l(\alpha,\beta|\underline{x})h(\alpha,\beta)}{\int_0^{\infty}\int_0^{\infty} l(\alpha,\beta|\underline{x})h(\alpha,\beta)\partial\alpha\partial\beta} \quad (22)$$

where,

$$k^{-1} = \int_0^\infty \int_0^\infty l(\alpha,\beta|\underline{x})\, h(\alpha,\beta)\partial\alpha\partial\beta \tag{23}$$

is a normalizing constant.

Substituting Equations (14) and (18) into Equation (22), we obtain:

$$p^*(\alpha,\beta|\underline{x}) = \frac{k\left(\frac{1}{\alpha\beta}\right)\prod_{i=1}^{n}\left(\frac{\beta}{\alpha}\left(\frac{x_i}{\alpha}\right)^{\beta-1} exp\left(-\left(\frac{x_i}{\alpha}\right)^{\beta}\right)\right)}{exp\left(-\left(\frac{u}{\alpha}\right)^{\beta}\right)-exp\left(-\left(\frac{v}{\alpha}\right)^{\beta}\right)} \tag{24}$$

So, from Equation (23), we get:

$$p^*(\alpha,\beta|\underline{x}) = \frac{\left(\frac{1}{\alpha\beta}\right)\prod_{i=1}^{n}\left\{\frac{\frac{\beta}{\alpha}\left(\frac{x_i}{\alpha}\right)^{\beta-1} exp\left(-\left(\frac{x_i}{\alpha}\right)^{\beta}\right)}{exp\left(-\left(\frac{u}{\alpha}\right)^{\beta}\right)-exp\left(-\left(\frac{v}{\alpha}\right)^{\beta}\right)}\right\}}{\int_0^\infty\int_0^\infty\left(\frac{1}{\alpha\beta}\right)\prod_{i=1}^{n}\left\{\frac{\frac{\beta}{\alpha}\left(\frac{x_i}{\alpha}\right)^{\beta-1} exp\left(-\left(\frac{x_i}{\alpha}\right)^{\beta}\right)}{exp\left(-\left(\frac{u}{\alpha}\right)^{\beta}\right)-exp\left(-\left(\frac{v}{\alpha}\right)^{\beta}\right)}\right\}\partial\alpha\partial\beta} \tag{25}$$

The posterior distribution of α and β is a ratio of functions which cannot be simplified to get a closed form result. Lindley's [16] approximation method has been used to solve such ratios (Howlader [17], Nassar [18], Xu [15], Jabarali et al. [19], Kim et al. [20], and Singh et al. [21]).

Lindley's Approximation Method: To obtain the Bayes estimator which is a ratio of two integrals, an approximation can be obtained by computing the posterior expectation of an arbitrary function $u(\alpha,\beta)$ as given by:

$$I(x) = E(u(\alpha,\beta)|x) = \frac{\int u(\alpha,\beta)\, exp\left(L(\alpha,\beta)+G(\alpha,\beta)\right)\partial(\alpha,\beta)}{\int exp\left(L(\alpha,\beta)+G(\alpha,\beta)\right)\partial(\alpha,\beta)} \tag{26}$$

where,

$u(\alpha,\beta) =$ a function of α and β only;

$L(\alpha,\beta) =$ log-likelihood function;

$G(\alpha,\beta) =$log of joint prior of α and β

According to Lindley's approximation, Equation (26) can be approximated asymptotically by:

$$\begin{aligned} I(x) = u(\hat{\alpha},\hat{\beta}) &+ \frac{1}{2}\left[(\hat{u}_{\alpha\alpha} + 2\hat{u}_{\alpha}\hat{\rho}_{\alpha})\hat{\sigma}_{\alpha\alpha} + (\hat{u}_{\beta\alpha} + 2\hat{u}_{\beta}\hat{\rho}_{\alpha})\hat{\sigma}_{\beta\alpha} + (\hat{u}_{\alpha\beta} + 2\hat{u}_{\alpha}\hat{\rho}_{\beta}) + (\hat{u}_{\beta\beta} + 2\hat{u}_{\beta}\hat{\rho}_{\beta})\hat{\sigma}_{\beta\beta}\right] \\ &+ \frac{1}{2}\left[(\hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{u}_{\beta}\hat{\sigma}_{\alpha\beta})(\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta} + \hat{L}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha} + L_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta})\right. \\ &\left. +(\hat{u}_{\alpha}\hat{\sigma}_{\beta\alpha} + \hat{u}_{\beta}\hat{\sigma}_{\beta\beta})(\hat{L}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta} + \hat{L}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha} + \hat{L}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta})\right] \end{aligned} \tag{27}$$

where,

$\hat{\alpha} =$ MLE of α; $\hat{\beta} =$ MLE of β; and

$\hat{u}_{\theta} = \frac{\partial u(\hat{\alpha},\hat{\beta})}{\partial\hat{\theta}}; \theta = \alpha,\beta;$ $\quad\quad$ $\hat{u}_{\theta\upsilon} = \frac{\partial^2 u(\hat{\alpha},\hat{\beta})}{\partial\hat{\theta}\,\partial\hat{\upsilon}}; \theta,\upsilon = \alpha,\beta;$

$\hat{L}_{\theta\upsilon\psi} = \frac{\partial^3 L(\hat{\alpha},\hat{\beta})}{\partial\hat{\theta}\,\partial\hat{\upsilon}\,\partial\hat{\psi}}; \theta,\upsilon,\psi = \alpha,\beta; \hat{\rho}_{\theta} = \frac{\partial G(\hat{\alpha},\hat{\beta})}{\partial\hat{\theta}}; \theta = \alpha,\beta.$

The elements $\hat{\sigma}_{\alpha\alpha}$, $\hat{\sigma}_{\beta\beta}$, $\hat{\sigma}_{\alpha\beta}$ and $\hat{\sigma}_{\beta\alpha}$ are the entries of $\sum = I^{-1}(\hat{\alpha}, \hat{\beta})$, where $I(\hat{\alpha}, \hat{\beta})$ is the observed information matrix evaluated at the MLEs $(\hat{\alpha}, \hat{\beta})$.

2.2.3.2.1 Bayes estimators of α and β under SELF:

Now, to obtain the Bayes estimator of α, consider

$$L(\alpha, \beta|\underline{x}) = n\, log\, \beta + (\beta - 1)\sum_{i=1}^{n} log(x_i) - n\beta\, log\, \alpha - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} - n\, log[e^{A} - e^{B}] \quad (28)$$

where,

$A = -\left(\frac{u}{\alpha}\right)^{\beta}$, and $B = -\left(\frac{v}{\alpha}\right)^{\beta}$

For $u(\hat{\alpha}, \hat{\beta}) = \alpha$; $G(\alpha, \beta) = ln(g(\alpha, \beta)) = -ln(\alpha) - ln(\beta)$, we have

$\hat{u}_{\alpha} = 1$; $\hat{u}_{\alpha\alpha} = 0$; and $\hat{u}_{\beta} = \hat{u}_{\beta\beta} = \hat{u}_{\beta\alpha} = \hat{u}_{\alpha\beta} = 0$

$\hat{\rho}_{\alpha} = -\left(\frac{1}{\alpha}\right)$; and $\hat{\rho}_{\beta} = -\left(\frac{1}{\beta}\right)$

Since α and β are independent, we have:

$\hat{\sigma}_{\alpha\beta} = \hat{\sigma}_{\beta\alpha} = 0$; $\hat{\sigma}_{\alpha\alpha} = -\frac{1}{\hat{L}_{\alpha\alpha}}$; and $\hat{\sigma}_{\beta\beta} = -\frac{1}{\hat{L}_{\beta\beta}}$

Now,
$$\hat{L}_{\alpha} = \frac{\partial L}{\partial \alpha} = -\frac{n\beta}{\alpha} + \frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} - \frac{n\beta\alpha^{-\beta-1}(u^{\beta}e^{A} - v^{\beta}e^{B})}{e^{A} - e^{B}} \quad (29)$$

and

$$\hat{L}_{\alpha\alpha} = \frac{\partial \hat{L}_{\alpha}}{\partial \alpha} = \frac{n\beta}{\alpha} - \frac{\beta(\beta+1)}{\alpha^{2}}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} - n\frac{N_{\alpha}D - ND_{\alpha}}{D^{2}} \quad (30)$$

where N, D and their first derivatives N_{α}, D_{α} are given by:

$N = \beta\alpha^{-\beta-1}(u^{\beta}e^{A} - v^{\beta}e^{B})$, $D = e^{A} - e^{B}$, $N_{\alpha} = \frac{\partial N}{\partial \alpha}$, $D_{\alpha} = \frac{\partial D}{\partial \alpha}$

Again,
$$\hat{L}_{\alpha\alpha\alpha} = \frac{\partial \hat{L}_{\alpha\alpha}}{\partial \alpha} = -\frac{2n\beta}{\alpha^{3}} + \frac{\beta(\beta+1)(\beta+2)}{\alpha^{3}}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} - n\frac{F_{\alpha}G - FG_{\alpha}}{G^{2}} \quad (31)$$

where F, G and their first derivatives F_{α}, G_{α} are given by:

$F = N_{\alpha}D - ND_{\alpha}$, $G = D^{2}$, $F_{\alpha} = \frac{\partial}{\partial \alpha}[N_{\alpha}D - ND_{\alpha}] = N_{\alpha\alpha}D - ND_{\alpha\alpha}$, $G_{\alpha} = \frac{\partial G}{\partial \alpha} = \frac{\partial}{\partial \alpha}[D^{2}] = 2DD_{\alpha}$

Similarly,

$$\hat{L}_{\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} log\, x_i - n\, log\, \alpha - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} log\left(\frac{x_i}{\alpha}\right) + n\frac{\left[\left(\frac{u}{\alpha}\right)^{\beta} log\left(\frac{u}{\alpha}\right)e^{A} - \left(\frac{v}{\alpha}\right)^{\beta} log\left(\frac{v}{\alpha}\right)e^{B}\right]}{e^{A} - e^{B}} \quad (32)$$

$$\hat{L}_{\beta\beta} = -\frac{n}{\beta^{2}} - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\left(log\left(\frac{x_i}{\alpha}\right)\right)^{2} + n\frac{M_{\beta}D - MD_{\beta}}{D^{2}} \quad (33)$$

where M, D and their first derivatives M_{β}, D_{β} are given by:

$$M = \left(\frac{u}{\alpha}\right)^{\beta} log\left(\frac{u}{\alpha}\right) e^{A} - \left(\frac{v}{\alpha}\right)^{\beta} log\left(\frac{v}{\alpha}\right) e^{B}, \qquad M_{\beta} = \frac{\partial M}{\partial \beta},$$

$$D = e^{A} - e^{B}, \quad D_{\beta} = \frac{\partial D}{\partial \beta}$$

and

$$\hat{L}_{\beta\beta\beta} = \frac{2n}{\beta^3} - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\left(log\left(\frac{x_i}{\alpha}\right)\right)^3 + n\frac{U_{\beta}V - UV_{\beta}}{V^2} \tag{34}$$

where U, V and their first derivatives U_{β}, V_{β} are given by:

$$U = M_{\beta}D - MD_{\beta}, \quad V = D^2, \quad U_{\beta} = \frac{\partial}{\partial \beta}\left(M_{\beta}D - MD_{\beta}\right) = M_{\beta\beta}D - MD_{\beta\beta}, \quad V_{\beta} = \frac{\partial V}{\partial \beta} = \frac{\partial D^2}{\partial \beta} = 2DD_{\beta}$$

The third-order mixed partial derivatives are:

$$\hat{L}_{\alpha\beta\beta} = \hat{L}_{\beta\alpha\beta} = \hat{L}_{\beta\beta\alpha} = \frac{1}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\left(\beta\left(log\left(\frac{x_i}{\alpha}\right)\right)^2 + 2\,log\left(\frac{x_i}{\alpha}\right)\right) - n\frac{\partial^3}{\partial\beta\partial\beta\partial\alpha} log\, D \tag{35}$$

$$\hat{L}_{\alpha\beta\alpha} = \hat{L}_{\beta\alpha\alpha} = \hat{L}_{\alpha\alpha\beta} = \frac{n}{\alpha^2} + \beta(\beta+1)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\left[\frac{1}{\alpha^2} log\left(\frac{x_i}{\alpha}\right) + \frac{2}{\alpha^3}\right] - n\frac{\partial^3}{\partial\alpha\partial\beta\partial\alpha} log\, D \tag{36}$$

By evaluating the u-terms, L-terms, and ρ-terms mentioned above at point $(\hat{\alpha}, \hat{\beta})$ and applying Equation (27), we obtain the Bayes estimator of α under the SELF is given as:

$$\hat{\alpha}_{BS} = \hat{\alpha} - \frac{1}{\hat{\alpha}}\hat{\sigma}_{\alpha\alpha} + \frac{1}{2}\left(\hat{L}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta} + \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}^2{}_{\alpha\alpha}\right) \tag{37}$$

Similarly, the Bayes estimator of β under the SELF is given as:

$$\hat{\beta}_{BS} = \hat{\beta} - \frac{1}{\hat{\beta}}\hat{\sigma}_{\beta\beta} + \frac{1}{2}\left(\hat{L}_{\alpha\alpha\beta}\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta} + \hat{L}_{\beta\beta\beta}\hat{\sigma}^2{}_{\beta\beta}\right) \tag{38}$$

2.2.3.3 Method of Moments Estimation by Cran

The moment estimates of the scale and shape parameters of left-right truncated two parameter Weibull distribution as proposed by Cran [22] are given as:

$$\boldsymbol{\hat{\alpha}_{cr}} = \frac{\boldsymbol{\bar{m}_1}}{\boldsymbol{\Gamma\left(1+\frac{1}{\hat{\beta}_{cr}}\right)}} \qquad \text{and} \qquad \boldsymbol{\hat{\beta}_{cr}} = \frac{\boldsymbol{ln(2)}}{\boldsymbol{ln(\bar{m}_1) - ln(\bar{m}_2)}}$$

where, $\bar{m}_r = \sum_{j=0}^{n-1}\left(1 - \frac{j}{n}\right)^r \left\{x_{(j+1)} - x_{(j)}\right\}$ with $x_{(0)} = 0$ and $x_{(j)}$ is the j^{th} ordered observation.

2.2.4 Probabilities of truncation limits

The probabilities of left and right classes and the probability of X_i to lie in the observation range are given as follows:

$$P(X_i \le u) = \int_0^u f(x_i|\alpha, \beta)\, dx_i \tag{39}$$

$$P(X_i \ge v) = \int_v^{\infty} f(x_i|\alpha, \beta)\, dx_i \tag{40}$$

and

$$P(u \le X_i \le v) = \int_u^v f(x_i|\alpha, \beta)\, dx_i \tag{41}$$

3. RESULTS

In all three surveys' data, it was found that the 'Difficulty' scores, which range from 0 to 40, the scores 0-1, and 31 and above were rare. So, it was decided to truncate these score classes. In fact, even after applying truncation, the total number of available responses was 1009 (out of a total 1020), 739 (out of 743) and 933 (out of 934), respectively, in three surveys.

Table 1 provides the frequencies and proportions of the participants of the three surveys in three severity bands, namely, the 'Normal', 'Borderline' and the 'Abnormal' bands of the observed 'Difficulty' scores of SDQ, both before and after truncation for the collected data.

Table 1: Frequency distribution and the proportions of participants in the three severity bands of 'Difficulty' scores based on the 'Observed' data (both truncated and complete) for all three surveys

Severity band	Survey 1		Survey 2		Survey 3	
	Complete (1020)	Truncated (1009)	Complete (743)	Truncated (739)	Complete (934)	Truncated (934)
Normal	678	674	527	524	488	487
	0.6647	0.6680	0.7093	0.7091	0.5225	0.5220
Borderline	196	196	121	121	224	224
	0.1922	0.1943	0.1629	0.1637	0.2398	0.2401
Abnormal	146	139	95	94	222	222
	0.1431	0.1378	0.1279	0.1272	0.2377	0.2379

The values were in agreement with the conditions prevailing at the times of the three surveys. Survey 3 was conducted just after the delta wave and had the highest proportion in "Abnormal" band among the three surveys.

Table 2 below provides the mean, median and variance of the observed 'Difficulty' scores for both the 'Complete' and 'Truncated' data of all three surveys. Although the number of participants has not reduced substantially after truncation, the effect of truncation on the variance of the data.

Table 2: Mean, median and variance for the 'Complete' and 'Truncated' data of all the Three Surveys

Moments	Survey 1		Survey 2		Survey 3	
	Complete	Truncated	Complete	Truncated	Complete	Truncated
Mean	13.7088	13.6194	12.7564	12.7767	15.1296	15.1296
Median	13	13	12	12	15	15
Variance	31.9573	28.8233	30.9419	29.9920	34.6709	34.6709

Due to a wide score range of total difficulty scores, it was decided to fit a continuous distribution to the data. However, the selection of an appropriate distribution should be on the basis of both the statistical measures and the interpretation aspect of the fit. Some of the common continuous distributions, namely Gamma, Weibull, Lognormal and Normal, were plotted along with the probability plot of the observed 'Difficulty' scores for all three surveys (Figure 1below)

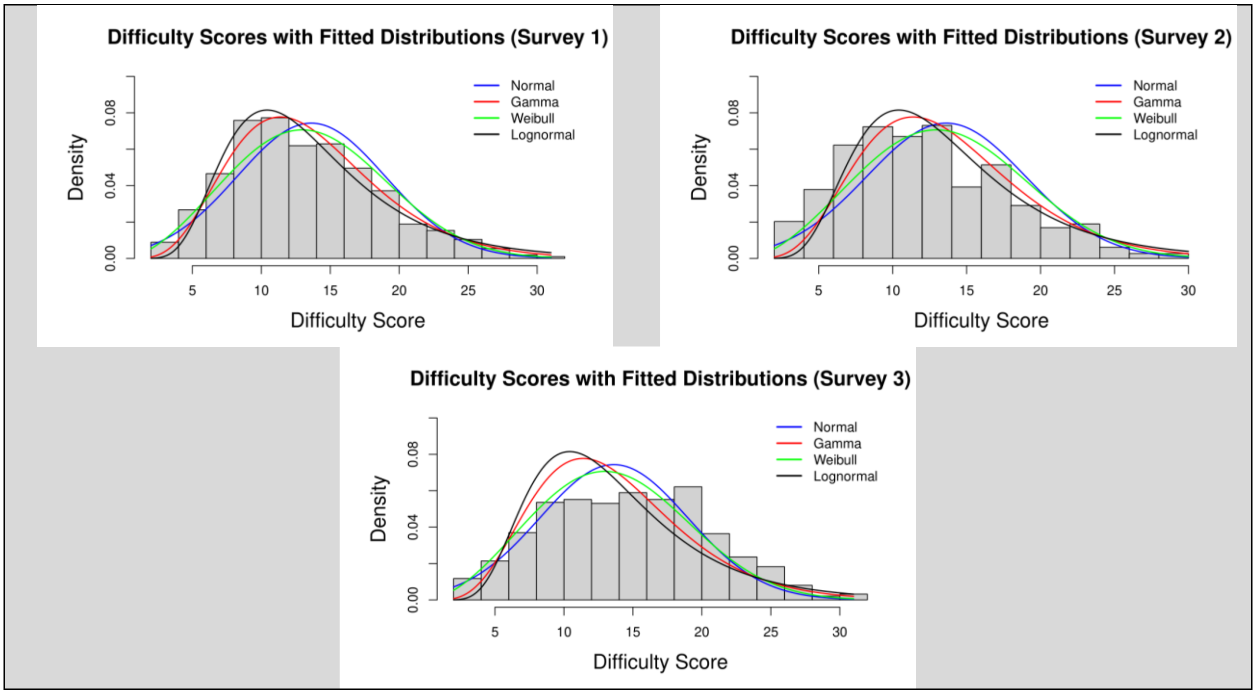

Figure 1: Distribution of Difficulty Scores for all Three Surveys

Although both Gamma and Weibull distributions were good fits for the first two surveys, for the third survey data, Weibull seemed to fit best. The findings were corroborated by AIC and BIC values of the fits. The results are presented in Table 3 below:

Table 3: AIC and BIC values of some continuous distributions fitted to the Difficulty Scores of all three surveys

Survey	Distribution	AIC Values	BIC Values	Selected Distribution
Survey 1	Normal	6431.32	6441.17	Weibull
	Gamma	6383.21	6393.06	
	Weibull	**6375.26**	**6385.12**	
	Lognormal	6473.05	6482.90	
Survey 2	Normal	4661.60	4670.82	Weibull
	Gamma	4645.18	4654.41	
	Weibull	**4622.65**	**4631.88**	
	Lognormal	4734.61	4743.83	
Survey 3	Normal	5970.19	5979.87	Weibull
	Gamma	6006.99	6016.67	
	Weibull	**5945.73**	**5955.41**	
	Lognormal	6117.06	6126.74	

The parameters of the Weibull distribution were obtained for the complete (using MLE, Cran and Bayesian estimates using Lindley approximation). The results are presented in the Table 4 below:

Table 4: The Estimates of the Parameters for the 'Complete' Data for all the Three Surveys

Model	Method	Survey 1		Survey 2		Survey 3	
		Scale $(\hat{\alpha})$	Shape $(\hat{\beta})$	Scale $(\hat{\alpha})$	Shape $(\hat{\beta})$	Scale $(\hat{\alpha})$	Shape $(\hat{\beta})$
$\mathrm{WD}(\alpha, \beta)$	MLE	15.44011	2.572254	14.38936	2.445961	16.98565	2.790381
	Cran	15.42169	2.668297	14.38206	2.464552	16.98403	2.756065
	Bayesian	15.44857	2.570047	14.40127	2.442831	16.99347	2.787346

AIC and BIC values provide a distribution that is closest to the data among the class of tested distributions. Although a Weibull distribution was the one with the lowest AIC and BIC values, further, the Kolmogorov- Smirnov (K-S) test was used to check the fit. The results of K-S test for all three surveys are presented in Table 5 below:

Table 5: Kolmogorov–Smirnov Test Statistics and Approximate p-values for the Weibull Model Across the Three Surveys

Survey	K-S Statistic (D)	P-Value
Survey 1	0.0291	0.3093
Survey 2	0.0272	0.3544
Survey 3	0.0262	0.3804

In all three surveys, the p-values were much larger than 0.05, thus indicating that the Weibull distribution fitted the data well.

Using the estimates in Table 4, the frequencies and proportions in the three severity bands of “Difficulty” scores were estimated for all three surveys (Table 6):

Table 6: Estimated Frequencies and probabilities in three severity bands for ‘Complete’ data of all three surveys

Severity band	Survey 1 (1020 Participants)			Survey 2 (743 Participants)			Survey 3 (934 Participants)		
	MLE	Bayesian	Cran	MLE	Bayesian	Cran	MLE	Bayesian	Cran
Normal	617	617	617	498	497	498	473	473	475
	0.6048	0.6043	0.6049	0.6695	0.6687	0.6702	0.5068	0.5065	0.5084
Borderline	257	257	265	166	166	167	268	268	265
	0.25238	0.2523	0.2599	0.2238	0.2239	0.2248	0.2867	0.2864	0.2837
Abnormal	146	146	138	79	80	78	193	193	194
	0.1429	0.1434	0.1352	0.1067	0.1074	0.1050	0.2065	0.2071	0.2082

Next, fitting a left-right truncated Weibull distribution yielded the following parameters:

Table 7: The Estimates of the Parameters for the ‘Truncated’ Data for all the Three Surveys

Model	Method	Survey 1		Survey 2		Survey 3	
		Scale $(\hat{\alpha})$	Shape $(\hat{\beta})$	Scale $(\hat{\alpha})$	Shape $(\hat{\beta})$	Scale $(\hat{\alpha})$	Shape $(\hat{\beta})$

LRTWD(α,β)	MLE	15.30073	2.663299	14.34243	2.416114	17.1089	2.698049
	Cran	15.30082	2.771571	14.3991	2.507329	16.99865	2.766625
	Bayesian	15.30468	2.663163	14.34789	2.416001	17.11353	2.698362

All three methods of estimation yielded estimates of the scale and shape parameters, which are fairly close to each other. Only the Cran estimates are slightly different in the third survey. The three truncated models were subject to the Kolmogorov-Smirnov test for goodness of fit. All three methods were equally good.

Table 8: Estimated Frequencies and probabilities in three severity bands for 'Truncated' data of all three surveys

Band	**Survey 1 (1009 Participants)**			**Survey 2 (739 Participants)**			**Survey 3 (933 Participants)**		
	MLE	**Bayesian**	**Cran**	**MLE**	**Bayesian**	**Cran**	**MLE**	**Bayesian**	**Cran**
Normal	619	619	618	497	497	495	473	473	475
	0.6133	0.6130	0.6122	0.6724	0.6721	0.6701	0.5073	0.5070	0.5095
Borderline	260	260	268	164	164	169	262	262	267
	0.2579	0.2580	0.2661	0.2216	0.2217	0.2284	0.2802	0.2802	0.2860
Abnormal	130	130	123	78	78	75	198	198	191
	0.1288	0.1290	0.1217	0.1059	0.1062	0.1015	0.2125	0.2127	0.2044

Finally, the moments were compared for all three surveys, both for the complete and the truncated data, obtained empirically and through a fitted model. As can be seen from Table 9 below, truncation has hardly any effect of mean, but it reduces variance in general.

Table 9: A comparison of empirical and estimated moments for the complete and truncated data for all the three surveys

Moments	**Survey 1**			
	Complete (1020)		**Truncated (1009)**	
	Empirical	**Estimated**	**Empirical**	**Estimated**
Mean	13.7088	13.709	13.6194	13.578
Median	13	13.38929	13	13.35345
Variance	31.9573	32.704	28.8233	**29.812**
Moments	**Survey 2**			
	Complete (743)		**Truncated (739)**	
	Empirical	**Estimated**	**Empirical**	**Estimated**
Mean	12.7564	12.760	12.7767	12.694
Median	12	12.38662	12	12.37134
Variance	30.9419	31.006	29.9920	**30.931**

Moments	Survey 3			
	Complete (934)		Truncated (933)	
	Empirical	Estimated	Empirical	Estimated
Mean	15.1296	15.122	15.1296	15.097
Median	15	14.89456	15	14.90428
Variance	34.6709	34.397	34.6709	34.993

Tables 10 and 11 compare the efficiencies of the estimates obtained using all three estimation techniques, based on the mean squared error (MSE) about the mean and the median, for both the complete and truncated models across all three surveys. Consistently, the MSEs were lower for truncated models.

Table 10: Comparison of the three estimation techniques* based on Mean Squared Error (MSE) Using Estimated Mean

Method	Survey 1		Survey 2		Survey 3	
	Complete	Truncated	Complete	Truncated	Complete	Truncated
MLE	31.92600	28.79642	30.90029	29.95835	34.81027	34.63484
MME (Cran)	31.92600	28.79489	30.90028	29.95172	34.81022	34.64129
Bayesian (Lindley)	31.92605	28.79618	30.90046	29.95770	34.81042	34.63459

*MLE, Bayesian, Cran

Table 11: Comparison of the three estimation techniques* based on Mean Squared Error (MSE) Using Estimated Median

Method	Survey 1		Survey 2		Survey 3	
	Complete	Truncated	Complete	Truncated	Complete	Truncated
MLE	32.02829	28.86569	31.03747	30.11611	34.85886	34.68471
MME (Cran)	31.99720	28.83368	31.03161	30.03955	34.87101	34.70173
Bayesian (Lindley)	32.02511	28.86410	31.03161	30.11288	34.85668	34.68292

*MLE, Bayesian, Cran

Finally, to compare the complete model with the truncated model, the table below enlists the empirical and the estimated frequencies in all three severity bands for all three surveys:

Table 12: A comparison of empirical and estimated frequencies (Cran's) for the complete and truncated data for all three surveys

Severity band	Survey 1			
	Complete (1020)		Truncated (1009)	
	Empirical	Estimated	Empirical	Estimated
Normal	678	617	674	618
Borderline	196	265	196	268
Abnormal	146	138	139	123

Severity band	Survey 2			
	Complete (743)		Truncated (739)	
	Empirical	Estimated	Empirical	Estimated
Normal	527	498	524	495
Borderline	121	167	121	169
Abnormal	95	78	94	75
Severity band	**Survey 3**			
	Complete (934)		**Truncated (933)**	
	Empirical	**Estimated**	**Empirical**	**Estimated**
Normal	488	475	487	475
Borderline	224	265	224	267
Abnormal	222	194	222	191

4. DISCUSSION

In the case of psychological data, the data is essentially qualitative in nature (as is collected using a Likert scale). The second characteristic of such data is the rarity of an unhealthy individual with all problems at extreme severity levels or a completely healthy individual with no problems at all. This will lead to almost essential restriction of the data domain and hence demands truncation of data. In the present study, data were collected on young individuals in Covid-19 pandemic period through the SDQ. Although the pandemic had severely impacted almost everyone and young adults were more affected, still, the number of those having a very high 'Difficulty' score was found to be so low that it was decided to truncate the data for higher score ranges (31 and more) for analytical purposes. Only a few were found to have a score of 0 or 1, so these data classes were also truncated.

The rationale for modelling qualitative data by a continuous parametric distribution is that the total "Difficulty" scores are obtained by adding the five) scores of each item in four scales range from 0-40 [8] and are treated as quantitative data. These scores are then assigned to the respondents.

One of the objectives of the study was to compare the three estimation techniques, namely, the Maximum Likelihood estimation, the Method of Moments (Cran) and the Bayesian Estimation method, in the case of truncated models. For complete data, studies such as by Sabharwal et al. [19], and Bihea [3] are available where the three estimation methods have been compared. Whereas [19] used standard error and power function to compare their results; [3] compared the results using mean squared error. However, in both these studies, the sample sizes were small, which were then validated using simulated data. In the present study, the three sample sizes (three surveys) were large (more than 700 in each case) and were independently obtained. The estimates obtained through three estimation techniques for the three data sets were consistent in estimating the frequencies (and probabilities) of three severity bands.

The three techniques of estimation, namely MLE (parametric), Cran (non-parametric) and Lindley (Bayesian) are used in different circumstances. However, for this data, the estimates obtained through the three techniques were consistent up to two places after the decimal. Further, the results were compared based on their Mean Squared Errors using the central tendency statistics, the mean and the median. In all three surveys and for all three techniques, the MSE of truncated data was found smaller than that of complete data, thus indicating an improvement with truncation. This was true for MSE using mean as well as median. Further, the MSEs using mean and the median were almost the same in all the cases, thus confirming that this qualitative data can be treated as quantitative data.

The data suggested that the extreme scores are so rare that a truncated model should be preferred over a complete model, as a complete model includes the information that generally does not exist in that domain. In the present study, this effect is visible from the reduction in variance, irrespective of how small a number was truncated (Table 9). Further, even if the extreme scores (particularly on the higher side) exist and are treated at the same level as the other scores (in case of the complete model), the model will not be able to capture the peculiarity and severity of the events. Such extreme cases should not be part of actual domain of a statistical distribution and should be handled separately. It is worthwhile to note that the data was collected on the seemingly healthy individuals.

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