International Journal of Computer Information Systems and Industrial Management Applications
ISSN 2150-7988 Volume 18(12s) 2026 pp. 467-487
https://doi.org/10.70917/ijcisim-2026-3917
Article

A Survival Signature Framework for Reliability and Component Importance Analysis of Coherent Systems with Cold Standby Components

Deepak[1], Garima Chopra[2,*], Kuldeep[3]
[1,2] Department of Mathematics, University Institute of Engineering and Technology, Maharshi Dayanand University, Rohtak, India.
[3]Department of Mathematics, Government College for Women, Karnal, India
[1]dkkr111@gmail.com
[2]garima.chopra@gmail.com
[3]kuldeepdahiya1641@gmail.com

Abstract: This study focuses on evaluating the system's reliability and identifying component criticality for a coherent system through the survival signature method. The Relative Importance Index is employed to identify critical components in the main system configuration before redundancy implementation. To enhance system reliability, each component is transformed into a subsystem containing active and independent non-identical standby components, where the standby components are analyzed both with and without degradation. Convolution-based recursive expressions are employed to evaluate the subsystem's reliability. Thereafter, system reliability and component importance measures are evaluated using the survival signature method for both the main system and the redundant system. A numerical illustration demonstrates that introducing cold standby redundancy, with and without degradation, significantly improves system reliability. It is also found that, even though the highly critical component of the original system remains influential, the relative criticality of subsystems under standby configurations changes over time, resulting in a shift in importance ranking. This redistribution of criticality highlights the effectiveness of the proposed redundancy strategy in improving system performance.
Keywords: reliability, component importance, survival signature, cold standby, degradation

1. Introduction

With the accelerating development of modern technologies, engineering systems have become more complex, interconnected, and highly automated than ever before. Present-day infrastructures, such as bridge networks, power grids, and communication systems, rely on numerous components working together simultaneously to ensure smooth and continuous operation. However, this increased complexity also makes these systems more vulnerable, as the failure of a single component can spread across the network and disrupt overall performance. Differences in component behaviour and varying failure patterns further make system operation and maintenance challenging. In such situations, reliability assessment becomes essential for evaluating system performance, understanding the impact of failures, and ensuring the safe and reliable functioning of complex engineering systems. There has been extensive discussion of traditional reliability analysis techniques, such as binary decision diagrams, fault trees, minimal cut and path sets, and reliability block diagrams in the introductory works of [3, 27, 30]. While these approaches are effective for relatively simple systems, they become computationally demanding when applied to large-scale systems comprising non-identical components [11]. To address this problem, Coolen and Coolen-Maturi [8] proposed the concept of survival signature, which distinguishes the structure of the system from component lifetime modeling and offers a more flexible approach for reliability assessment of complex systems. Eryilmaz and Tuncel [9] expanded its applicability to non-repairable multi-state systems having multi-state components. Survival signature-based Monte Carlo simulation approach for a standalone system with induced dependencies was extended in [14]. Furthermore, Huang et al. [18] utilized the concept of survival signatures in the analysis of general phased mission systems. Recently, a new algorithm was introduced to compute survival signatures through a weighted random sampling technique [38], which involves less computation time. In addition, Shi et al. [31] developed a graph-based neural network framework for estimating survival signatures and evaluating network reliability. Studies [16, 23, 25] also highlight the increasing research focus and progress in applying survival signatures across different disciplines.

The increasing need for high reliability in both competitive and safety-sensitive situations has intensified the importance of improving reliability and pinpointing essential components. The allocation of redundancy is among the most efficient strategies for improving system reliability. Several studies [1, 7, 19] have provided classical contributions to redundancy analysis. In this context, cold standby systems are especially attractive due to the minimised usage and cost associated with backup components, which remain inactive until needed. Earlier studies [13, 33] analyzed the reliability of cold standby systems, and more recently [29] examined the restart behavior of a coherent system with a single cold standby component. Hesampour et al. [17] highlighted both the theoretical and practical implications of the mixed redundancy strategy in the case of heterogeneous components. There is a possibility that standby components degrade even when not in use, which negatively impacts system performance and reduces the effectiveness of reliability gains. Wang and Coit [32] evaluated the reliability of the system through the employment of the degradation modeling approach to analyze the failure mechanism. The effect of degradation in standby components on the redundancy allocation problem is discussed in [33] and the reliability with a design optimization methodology considering the effects of dynamic degradation is discussed in [26]. Recent contributions [5, 37] integrate advanced degradation processes and imperfect switching behaviors. The authors demonstrate that degradation factors cannot be ignored when considering standby systems, especially for long-term operations. Wu et al. [35] performed an analytic derivation of reliability and lifetime assessment and applied Monte Carlo simulations to validate the fitting performance of the multi-gamma process in degradation.

In addition to improving system reliability, researchers have increasingly focused on assessing the importance of individual components within a system. Measures of component importance are essential for identifying system risks and guiding strategic decisions during design and maintenance. In 1969, the Birnbaum importance measure was first introduced [4] and further developed in [3], and more recent refinements were provided in [20, 39]. The concept of time-dependent and reliability-based importance analysis has also been discussed in [34]. Feng et al. [12] studied the Relative Importance Index (RII) as a technique for evaluating the significance of individual components in a system with various types of components. Additionally, Li et al. [21] explored an imprecise measure of component importance in a redundant system by considering swapping of components. A more recent study [36] focused on deriving the Birnbaum importance and Barlow–Proschan importance using the system's signature. Zheng [38] developed an index for assessing component importance in a complex system consisting of independent and non-identically distributed (INID) components, utilizing the survival signature. Furthermore, Chen and Feng [6] investigated a two-level approach to preventive maintenance based on a group importance metric. Mutar and Hassan [26] introduced the Birnbaum Importance and Improvement Potential measures for multi-state systems with binary components, applying the survival signature approach.

The literature on survival signatures, importance measures, and standby systems with degradation is extensive, but little research has been performed on how cold standby redundancy affects the shifting of component criticality over time in complex system structures. In particular, the dynamic shifts in component importance after redundancy implementation have not been thoroughly analyzed, especially for INID cold standby configurations. To address this gap, the present study proposes an approach that integrates survival signatures with cold standby strategies while incorporating degradation effects, with a focus on reliability improvement and shifts in component criticality. The bridge system is first analyzed to identify its critical components, and then cold standby components are incorporated into the system to enhance its reliability. This means that active components are transformed into subsystems, each comprising additional independent and non-identical cold standby components. In the original configuration, the analysis focuses on component criticality, whereas after the introduction of redundancy, each component is transformed into a subsystem, and the corresponding subsystem criticality is evaluated.

2. Definitions and Notation

The Survival signature, $\Phi(l)$, of a system consisting of m independent and identically distributed (IID) components, is the probability that the system is operational, given that l of the components are functioning. The operational state of the i^{th} $(i \in 1, 2 \ldots, m)$ component at time t is denoted by x_i, where $x_i = 0$ indicates a non-functional state and $x_i = 1$ signifies functionality. There exist $\binom{m}{l}$ unique state vectors, denoted by $x = (x_1, x_2, \ldots, x_i, \ldots, x_m)$, that meet the condition $\sum_{i=1}^{m} x_i = l$ with precisely l operational components, which are collectively referred to as s_l. The survival signature [8] of the considered system can be expressed mathematically as follows:

$$\Phi(l) = \frac{\sum_{x \epsilon s_l} \phi(x)}{|s_l|} = \binom{m}{l}^{-1} \sum_{x \epsilon s_l} \phi(x), \tag{2.1}$$

where the structure function, represented as $\phi(x)$, takes the value of 1 when the system is operational and 0 when it is not. If $R(t)$ denotes the survival function of 'n' IID components, then system's survival function $R_s(t)$ can be defined as:

$$R_s(t) = \sum_{l=0}^{m} \Phi(l) \binom{m}{l} [1 - R(t)]^{m-l} [R(t)]^l. \tag{2.2}$$

Suppose there is an m-order coherent system consisting of n different types, with each type k $(1 \le k \le n)$ having m_k components, totalling m components across all types. It is assumed that the components of each type share an identical failure time distribution. The system state can be described by the vector $x = (x^1, x^2, \ldots, x^n)$ where $x^k = \left(x_1^k, x_2^k, \ldots, x_{m_k}^k\right)$ represents the states of the m_k components of type k. The survival signature, denoted as $\Phi(l_1, l_2, \ldots, l_n)$, is the probability that the system is functioning given that exactly l_k components of type k are functioning, with $l_k \in \{0, 1, \ldots, m_k\}$ for each k. Additionally, for each type k, there are $\binom{m_k}{l_k}$ distinct state vectors x^k where exactly l_k components are operational ($x_s^k = 1$), satisfying the condition $\sum_{s=1}^{m_k} x_s^k = l_k$. The set of all these vectors is represented as $s_{l_1, l_2, \ldots, l_n}$. Therefore, the survival signature [8], $\Phi(l_1, l_2, \ldots, l_n)$ is articulated as follows:

$$\Phi(l_1, l_2, \ldots., l_n) = \left[\prod_{k=1}^{n} \binom{m_k}{l_k}^{-1}\right] \sum_{x \epsilon s_{l_1, l_2, \ldots, l_n}} \phi(x). \tag{2.3}$$

Furthermore, the system reliability can be obtained as follows:

$$R_s(t) = \sum_{l_1=0}^{m_1} \ldots. \sum_{l_n=0}^{m_n} \left[\Phi(l_1, l_2, \ldots., l_n) \prod_{k=1}^{n} \binom{m_k}{l_k} \; [1 - R_k(t)]^{m_k - l_k} [R_k(t)]^{l_k}\right], \tag{2.4}$$

where $R_k(t)$ be the type k components' reliability.

We define the restricted survival signature, $\Phi_i(l)$, as the probability that both the system and its i^{th} component function, conditioned on exactly l components being operational. This concept is derived from the survival signature and is restricted to coherent systems where $\Phi_i(l)$ is an increasing function for each i. Additionally, $\Phi_i(l) = 0$ if no components work and $\Phi_i(m) = 1$ for all $i \in 1,2, \ldots, m$. Here, $\bar{x}_i = (x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_m)$ denotes a state vector where the i^{th} component is operational, and the family of all such vectors is represented as $S_{l,i}$. Under the assumption of IID failure times, each state vector is equally probable. Consequently, the restricted survival signature is defined as

$$\Phi_i(l) = \binom{m}{l}^{-1} \sum_{\bar{x}_i \epsilon S_{l,i}} \phi(\bar{x}_i).$$

Similarly, the restricted survival signature $\bar{\Phi}_i(l)$, when the i^{th} component has failed takes the form

$$\bar{\Phi}_i(l) = \binom{m}{l}^{-1} \sum_{\underline{x_i} \epsilon \bar{S}_{l,i}} \phi(\underline{x_i}),$$

where the vector $\underline{x_i} = (x_1, x_2, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_m)$ be a state vector in which component $'i'$ is not working, and the set of all such vectors is denoted as set $\bar{S}_{l,i}$.

For m -order coherent systems with the n types of components, the restricted survival signature, $\Phi_i(l_1, l_2, \dots, l_n)$, is defined as the probability that the system functions and that its i^{th} component is functioning, conditional on exactly l_k components of type k being operational, for $k = 1,2, \dots, n$. Closely related to the concept of the survival signature, $\Phi_i(l_1, l_2, \dots, l_n)$ is an increasing function for each i in coherent systems. In particular

$$\Phi_i(0, 0, \dots, 0) = 0 \; and \; \Phi_i(m_1, m_2, \dots, m_n) = 1$$

for all $i \in \{1,2, \dots, m\}$.

Assuming the i^{th} component is the p^{th} component of type k, the state vector, denoted as $\bar{x}_i = (x^1, x^2, \dots, x^n)$, with$x^k = \left(x_1^k, x_2^k, \dots, x_{p-1}^k, 1, x_{p+1}^k, \dots, x_{m_k}^k\right)$ represents the states in which the i^{th} component is operational. The set of state vectors satisfying $x_p^k = 1$ is denoted as $S^i_{l_1, l_2, \dots, l_n}$. The restricted survival signature is then defined as:

$$\Phi_i(l_1, l_2, \dots, l_n) = \left(\prod_{k=1}^{n} \binom{m_k}{l_k}^{-1}\right) \sum_{\bar{x}_i \epsilon S^i_{l_1, l_2, \dots, l_n}} \phi(\bar{x}_i). \tag{2.5}$$

Similarly, if the i^{th} component is in a failed state, the restricted survival signature, $\bar{\Phi}_i(l_1, l_2, \dots, l_K)$, can be stated as:

$$\bar{\Phi}_i(l_1, l_2, \dots, l_n) = \left(\prod_{k=1}^{n} \binom{m_k}{l_k}^{-1}\right) \sum_{\underline{x_i} \epsilon \bar{S}^i_{l_1, l_2, \dots, l_n}} \phi(\underline{x_i}), \tag{2.6}$$

where $\underline{x_i} = (x^1, x^2, \dots, x^n)$ is the state vector with $x^k = \left(x_1^k, x_2^k, \dots, x_{p-1}^k, 0, x_{p+1}^k, \dots, x_{m_k}^k\right)$, collectively represented by the set $\bar{S}^i_{l_1, l_2, \dots, l_n}$. It can also be represented in terms of the number of path sets that contain precisely l_k functioning components, denoted by $r_{l_1, l_2, \dots, l_n}(m_1, m_2, \dots, m_n)$ and the number of path sets that include the functioning i^{th} component with exactly l_k operational units of type k, denoted by ${r^i}_{l_1, l_2, \dots, l_n}(m_1, m_2, \dots, m_n)$ as follows:

$$\bar{\Phi}_i(l_1, l_2, \dots, l_n) = \frac{r_{l_1, l_2, \dots, l_n}(m_1, m_2, \dots, m_n) - {r^i}_{l_1, l_2, \dots, l_n}(m_1, m_2, \dots, m_n)}{\prod_{k=1}^{n} \binom{m_k}{l_k}}$$

This simplifies to

$$\Phi(l_1, l_2, \dots, l_n) = \Phi_i(l_1, l_2, \dots, l_n) + \bar{\Phi}(l_1, l_2, \dots, l_n) \tag{2.7}$$

Moreover, letting T and T_i denote the lifetimes of the system and the i^{th} component, respectively, the probability that the system operates while the i^{th} component is functioning (failing) can be written by the following Equations 2.8 and 2.9:

$$P(T > t, T_i > t) = \sum_{l_1=0}^{m_1} \dots . \sum_{l_n=0}^{m_n} \left[\Phi_i(l_1, l_2, \dots ., l_n) \prod_{k=1}^{n} \binom{m_k}{l_k} \; [1 - R_k(t)]^{m_k - l_k} [R_k(t)]^{l_k} \right], \tag{2.8}$$

and

$$P(T > t, T_i \le t) = \sum_{l_1=0}^{m_1} \dots . \sum_{l_n=0}^{m_n} \left[\overline{\Phi}_i(l_1, l_2, \dots ., l_n) \prod_{k=1}^{n} \binom{m_k}{l_k} \; [1 - R_k(t)]^{m_k - l_k} [R_k(t)]^{l_k} \right]. \tag{2.9}$$

3. Main Results

3.1. System with cold standby components

Let the m-order coherent system have n distinct types. For each type, denoted by k, where $1 \le \; k \le n$, there are m_k components, and the sum of all m_k's equals m, i.e., $\sum_{k=1}^{n} m_k = m$. Consider P_k is the subsystem constructed by one active component of type k and $r_k - 1$ INID cold standby components with perfect switching. The system is now a combination of these subsystems P_k containing one principal component with $r_k - 1$ standby components.

Theorem 3.1: Let $T_{k,1}$ and $T_{k,j+1}$ for $j = 1, 2, \dots, r_k - 1$, denote the lifetimes of the type k active component and $(r_k - 1)$ standby components, respectively. Let $S_{k,j}$ be the sum of the failure times of the first j components of the subsystem P_k. Then the survival function of the subsystem P_k, denoted by $R_k^C(t)$, is given by

$$R_k^C(t) = R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t R_{k,j+1}(t-x) f_k^{(j)}(x)\, dx,$$

where the function $f_k^{(j)}(x)$ is the probability density function (pdf) of the sum of failure times of the first j components. Here, $R_k(t)$ denotes the type k active component's reliability in the subsystem P_k, and $R_{k,j+1}(t)$ represents the reliability function of cold standby components $\forall\, j \in \{1,2, \dots, r_k - 1\}$.

Proof: Let the subsystem P_k has one active and one cold standby component (i.e., $j = 1$). If $T_k^{C_{1,1}}$ be the lifetime of the subsystem P_k then $T_k^{C_{1,1}} = T_{k,1} + T_{k,2} = S_{k,2} \; and \; S_{k,1} = T_{k,1}$. The subsystem reliability, denoted by $R_k^{C_{1,1}}(t)$ is as follows

$$R_k^{C_{1,1}}(t) = P\{T_{k,1} > t\} \cup \{S_{k,1} \le t, S_{k,2} > t\}$$

i.e., $$R_k^{C_{1,1}}(t) = R_k(t) + \int_0^t R_{k,2}(t-x)\, f_k^{(1)}(x)\; dx.$$

The integral incorporates the failure time x of the active component, and ensures that the standby component operates as the active unit for the remaining time $t - x$. Thus, the hypothesis is satisfied in the case of one standby component ($j = 1$).

Assume that the hypothesis holds for one active and up to $r_k - 2$ cold standby components. The survival function of the subsystem P_k, denoted by $R_k^{C_{1,r_k-2}}(t)$, is

$$R_k^{C_{1,r_k-2}}(t) = R_k(t) + \sum_{j=1}^{r_k-2} \int_0^t R_{k,j+1}(t-x)\, f_k^{(j)}(x)\; dx$$

(3.1)

We now expand the subsystem P_k with an additional cold standby component with a lifetime T_{k,r_k}, then the subsystem's lifetime be the sum of all lifetimes of the active and cold standby components, i.e., $T_k^C = T_{k,1} + T_{k,2} + \cdots T_{k,r_k-1} + T_{k,r_k}$. The subsystem P_k survives beyond time t if either the first $r_k - 1$ components together survive beyond t or the first $r_k - 1$ components do not work at time $x \leq t$ and $r_k{}^{th}$ component survives for the remaining time $t - x$. Let $S_{k,r_k-1} = T_{k,1} + T_{k,2} + \cdots + T_{k,r_k-1}$ and $S_{k,r_k} = S_{k,r_k-1} + T_{k,r_k}$. Therefore, the subsystem's reliability, denoted as $R_k^C(t)$, can be derived as:

$$R_k^C(t) = P\left(S_{k,r_k} > t\right) = P\left(S_{k,r_k-1} + T_{k,r_k} > t\right)$$

Then the event $\{S_{k,r_k} + T_{k,r_k} > t\}$ can be transformed into $\left\{S_{k,r_k-1} > t\right\} \cup \left\{S_{k,r_k-1} \leq t, S_{k,r_k} > t\right\}$. So,

$$R_k^C(t) = R_k^{C_{1,r_k-2}}(t) + P\left(S_{k,r_k-1} \leq t, S_{k,r_k} > t\right) \tag{3.2}$$

Also,

$$P\left(S_{k,r_k-1} \leq t, S_{k,r_k} > t\right) = \int_0^t R_{k,r_k}(t-x)\ f_k^{(r_k-1)}(x)\ dx \tag{3.3}$$

By substituting the induction hypothesis from equation (3.1) into equation (3.2), we get

$$R_k^C(t) = R_k(t) + \sum_{j=1}^{r_k-2} \int_0^t R_{k,j+1}(t-x)\, f_k^{(j)}(x)\ dx\ +\ P\left(S_{k,r_k-1} \leq t, S_{k,r_k} > t\right)$$

From equation (3.3), we obtain

$$R_k^C(t) = R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t R_{k,j+1}(t-x)\, f_k^{(j)}(x)\ dx \tag{3.4}$$

Using the "principle of mathematical induction", the result is valid for one active and $r_k - 1$ cold standby components. This completes the proof.

The system reliability after adding $r_k - 1$ INID cold standby components to the type k active component can be computed using equations (2.4) and (3.4) as follows:

$$R_s(t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_n=0}^{m_n} \left[\Phi(l_1, l_2, \ldots, l_n) \prod_{k=1}^{n} \binom{m_k}{l_k} \left[1 - \left(R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t R_{k,j+1}(t-x)\, f_k^{(j)}(x)\ dx\right)\right]^{m_k-l_k} \left[R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t R_{k,j+1}(t-x)\, f_k^{(j)}(x)\ dx\right]^{l_k}\right].$$

(3.5)

Assuming all components in the subsystem P_k are IID i.e., $R_k(t) = R_{k,j+1}(t) \quad \forall\, j \in \{1, 2, \dots, r_k - 1\}$, equation (3.5) can be rewritten as:

$$R_s(t) = \sum_{l_1=0}^{m_1} \dots . \sum_{l_n=0}^{m_n} \left[\Phi(l_1, l_2, \dots ., l_n) \prod_{k=1}^{n} \binom{m_k}{l_k} \left[1 - \left(R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t R_k(t-x)\, f_k^{(j)}(x)\ dx \right) \right]^{m_k - l_k} \left[R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t R_k(t-x)\, f_k^{(j)}(x)\ dx \right]^{l_k} \right].$$

(3.6)

3.2. System with degrading cold standby components

Degrading cold standby components can significantly impact the system reliability, even when not actively in use, these components experience wear and tear, which can lead to unexpected failures. As a result, accurately modelling their degradation is essential for the system performance. Consider the degrading cold standby components, with reliability $\tilde{R}_{k,j+1}(x)$, $j \in \{1, 2, \dots, r_k - 1\}$ while in the standby state for time x.

Theorem 3.2: Let $T_{k,1}$ be the lifetime of the active component of type k and $T_{k,j+1}$ be the lifetimes of $r_k - 1$ degrading standby components for $j \in \{1, 2, \dots, r_k - 1\}$ in the subsystem P_k. Also, let $S_{k,j}$ be the sum of the first j components' failure times in the subsystem P_k. Then the survival function of the subsystem P_k, denoted by $R_k^D(t)$, is

$$R_k^D(t) \ = R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t \tilde{R}_{k,j+1}(x)\, R_{k,j+1}(t -\ x)\, \tilde{f}_k^{(j)}(x)\ dx,$$

where $\tilde{f}_k^{(j)}(x)$ is used to represent the pdf for the j^{th} failure occurrence of subsystem P_k, considering the degrading effect on standby components. $R_k(t)$ denotes the reliability of the type k active component in the subsystem P_k. Further $R_{k,j+1}(t)$ and $\tilde{R}_{k,j+1}(t)$ represent the reliabilities of the cold standby components without and with degradation respectively $\forall\, j \in \{1,2, \dots, r_k - 1\}$.

Proof: Let's consider the subsystem P_k, which has one active component and one degrading standby component (i.e., $j = 1$). Let $T_k^{D_{1,1}}$ be the lifetime of the subsystem P_k, then $T_k^{D_{1,1}} = T_{k,1} + T_{k,2}$, $S_{k,1} = \quad T_{k,1}$ and $S_{k,2} = \quad T_{k,1} + T_{k,2}$. If the active component fails at time x, then the standby component is triggered, and its reliability is given by $\tilde{R}_{k,2}(x) \quad R_{k,2}$ (t - x), for $t \geq x$. Therefore,

$$P\left(S_{k,1} \leq t, S_{k,2} > t\right) = \int_0^t \tilde{R}_{k,2}(x)\, R_{k,2}(t - x)\, \tilde{f}_k^{(1)}(x)\ dx,$$

where $f_k^{(1)}(x)$ is identical to $\tilde{f}_k^{(1)}(x)$, is the pdf of the active component for time x. Following the induction basis as in Theorem 3.1, the subsystem reliability $R_k^D(t)$ is given by

$$R_k^D(t) = R_k(t) + \int_0^t \tilde{R}_{k,2}(x)\, R_{k,2}(t - x)\, f_k^{(1)}(x)\ dx.$$

Thus, the hypothesis is satisfied in the case of one degrading standby component ($j = 1$).

Consider the hypothesis to be true for one active and up to $r_k - 2$ degrading cold standby components. Then the survival function of the subsystem P_k, denoted by $R_k^{D_{1,r_k-2}}(t)$, is given by

$$R_k^{D_{1,r_k-2}}(t) = R_k(t) + \sum_{j=1}^{r_k-2} \int_0^t \tilde{R}_{k,j+1}(x)\, R_{k,j+1}(t - x)\, \tilde{f}_k^{(j)}(x)\ dx. \tag{3.8}$$

We now expand the subsystem P_k by adding an additional cold standby component with lifetime T_{k,r_k}. Then $T_k^D = \sum_{j=1}^{r_k} T_{k,j}$. The subsystem P_k survives beyond time t if either the first $r_k - 1$ components together survive beyond t or the first $r_k - 1$ components do not work at time $x \leq t$ and ${r_k}^{th}$ component survives for the remaining time $t - x$. Let $S_{k,r_k-1} = T_{k,1} + T_{k,2} + \cdots + T_{k,r_k-1}$ and $S_{k,r_k} = S_{k,r_k-1} + T_{k,r_k}$. Proceeding similarly to the induction step in Theorem 3.1, the subsystem's reliability, $R_k^D(t)$, can be derived as follows

$$R_k^D(t) = R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t \tilde{R}_{k,j+1}(x)\, R_{k,j+1}(t - x)\, \tilde{f}_k^{(j)}(x)\ dx. \tag{3.11}$$

Using the "principle of mathematical induction", the hypothesis is valid for one active and $r_k - 1$ degrading cold standby components. This completes the proof.

Remark: If $\tilde{f}_k^{(j)}(x)$ be the j-fold convolution of the degrading cold standby component lifetime, then following relation can be found

$$\tilde{f}_k^{(j)}(x) = \int_0^x \tilde{R}_{k,j}(u)\, f_{k,j}(x - u)\, f_k^{(j-1)}(u)\ du,$$

where $f_{k,j}(x - u), j \in \{2, 3, ..., r_k - 1\}$ is the pdf of the cold standby components for the time $(x - u)$ and the function $f_k^{(j-1)}(u)$ is the pdf of sum of first $j - 1$ components' failure times.

By incorporating $r_k - 1$ INID degrading cold standby components into the type k active component, the system reliability can be determined using equations (2.4) and (3.11) as:

$$R_s(t) = \sum_{l_1=0}^{m_1} \dots \sum_{l_n=0}^{m_n} \left[\Phi(l_1, l_2, \dots, l_n) \prod_{k=1}^{n} \binom{m_k}{l_k} \left[1 - \left(R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t \tilde{R}_{k,j+1}(x)\, R_{k,j+1}(t-x)\, \tilde{f}_k^{(j)}(x)\; dx \right) \right]^{m_k-l_k} \left[R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t \tilde{R}_{k,j+1}(x)\, R_{k,j+1}(t-x)\, \tilde{f}_k^{(j)}(x)\; dx \right]^{l_k} \right]. \tag{3.12}$$

In the case where the degrading cold standby components are iid, assuming reliability $\tilde{R}_k(t)$, the Equation (3.12) becomes:

$$R_s(t) = \sum_{l_1=0}^{m_1} \dots \sum_{l_n=0}^{m_n} \left[\Phi(l_1, l_2, \dots, l_n) \prod_{k=1}^{n} \binom{m_k}{l_k} \left[1 - \left(R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t \tilde{R}_k(x)\, R_k(t-x)\, \tilde{f}_k^{(j)}(x)\; dx \right) \right]^{m_k-l_k} \left[R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t \tilde{R}_k(x)\, R_k(t-x)\, \tilde{f}_k^{(j)}(x)\; dx \right]^{l_k} \right], \tag{3.13}$$

Moreover, deriving a closed-form expression for Equation (3.13) involves considerable difficulty. Therefore, an approximation to the system reliability, denoted by $\tilde{R}_s(t)$ [33], is considered and is given as follows:

$$R_s(t) \geq \tilde{R}_s(t) = \sum_{l_1=0}^{m_1} \dots \sum_{l_n=0}^{m_n} \left[\Phi(l_1, l_2, \dots, l_n) \prod_{k=1}^{n} \binom{m_k}{l_k} \left[1 - \left(R_k(t) + \sum_{j=1}^{r_k-1} \left(\tilde{R}_k(t)\right)^{j+1} \int_0^t R_k(t-x)\, f_k^{(j)}(x)\; dx \right) \right]^{m_k-l_k} \left[R_k(t) + \sum_{j=1}^{r_k-1} \left(\tilde{R}_k(t)\right)^{j+1} \int_0^t R_k(t-x)\, f_k^{(j)}(x)\; dx \right]^{l_k} \right]. \tag{3.14}$$

The equation (3.14) holds since each replacement of a failed component will have to work up to the last standby component, which implies that $\tilde{R}_k(x) \geq \tilde{R}_k(t)\ \forall\ x \leq t$.

Assume that all components in the P_k follow an exponential distribution, i.e., $f_k(x) = \lambda_k e^{-\lambda_k x}$, where λ_k represents the failure rate of the type k components. Then, the j-fold convolution $f_k^{(j)}(x)$ is obtained as

$$f_k^{(j)}(x) = \frac{\lambda_k e^{-\lambda_k x}(\lambda_k x)^{j-1}}{(j-1)!}.$$

(3.15)

Using equations (3.14) and (3.15), the approximated survival function of the considered system can be stated as

$$\tilde{R}_s(t) = \sum_{l_1=0}^{m_1} \dots \sum_{l_n=0}^{m_n} \left[\Phi(l_1, l_2, \dots, l_n) \prod_{k=1}^{n} \binom{m_k}{l_k} \left[1 - \left(e^{-\lambda_k t} + \sum_{j=1}^{r_k-1} \left(\tilde{R}_k(t)\right)^{j+1} \int_0^t R_k(t-x) \frac{\lambda_k e^{-\lambda_k x}(\lambda_k x)^{j-1}}{(j-1)!} dx \right) \right]^{m_k - l_k} \left[e^{-\lambda_k t} + \sum_{j=1}^{r_k-1} \left(\tilde{R}_k(t)\right)^{j+1} \int_0^t R_k(t-x) \frac{\lambda_k e^{-\lambda_k x}(\lambda_k x)^{j-1}}{(j-1)!} dx \right]^{l_k} \right].$$

(3.16)

The survival analysis incorporating cold standby components subjected to degradation was studied in [22] using the degradation path of the component. The degradation path is given by $\eta(t) = \phi + \Theta t$, where ϕ represents initial degradation, and Θ represents the degradation rate. Let D be the critical degradation limit, and suppose that the component fails if the degradation exceeds this limit. If the failure limit is defined as $D = \phi + \Theta \hat{T}$, then $\hat{T} = \frac{D-\phi}{\Theta}$. The survival function of $\hat{T}$ in the standby state can therefore be written as

$$\tilde{R}_{\hat{T}}(t) = P(\hat{T} > t) = 1 - P\left(\Theta \geq \frac{D-\phi}{t}\right).$$

Let Θ_k be the degradation rate of type k cold standby components, following the Weibull distribution with parameters (α_k, β_k). The survival function, denoted by $\tilde{R}_k(t)$, is given by

$$\tilde{R}_k(t) = 1 - P\left(\Theta_k \geq \frac{D-\phi}{t}\right) = 1 - e^{-\left(\frac{D-\phi}{\alpha_k t}\right)^{\beta_k}}.$$

Assuming that all parameter in the degradation model is known. i.e., $\frac{\alpha_k}{D-\phi} = \alpha'_k$, then

$$\tilde{R}_k(t) = 1 - e^{-\left(\frac{1}{\alpha'_k t}\right)^{\beta_k}}$$

From equation (3.16), the approximated survival function $\tilde{R}_s(t)$ can be represented as

$$\tilde{R}_s(t) = \sum_{l_1=0}^{m_1} \dots \sum_{l_n=0}^{m_n} \left[\Phi(l_1, l_2, \dots, l_n) \prod_{k=1}^{n} \binom{m_k}{l_k} \left[1 - \left(e^{-\lambda_k t} + \sum_{j=1}^{r_k-1} \left(1 - e^{-\left(\frac{1}{\alpha'_k t}\right)^{\beta_k}} \right)^{j+1} \frac{e^{-\lambda_k t}(\lambda_k t)^j}{(j)!} \right) \right]^{m_k - l_k} \left[e^{-\lambda_k t} + \sum_{j=1}^{r_k-1} \left(1 - e^{-\left(\frac{1}{\alpha'_k t}\right)^{\beta_k}} \right)^{j+1} \frac{e^{-\lambda_k t}(\lambda_k t)^j}{(j)!} \right]^{l_k} \right]. \quad (3.17)$$

4. Relative importance Index (RII)

Feng et al. [12] defined the term RII for identifying the criticality of the *i*

$$RII_i = P\{T > t \,|T_i > t\} - P \quad \{T > t|T_i \leq t\}.$$

From equations (2.8) and (2.9), the RII of the i^{th} component is

$$RII_i \quad = \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m_2} \dots \sum_{l_n=0}^{m_n} \left\{ \left(\frac{\Phi_i(l_1, l_2, \dots, l_n)}{R_i(t)} \right) - \left(\frac{\overline{\Phi}_i(l_1, l_2, \dots, l_n)}{1 - R_i(t)} \right) \right\} \prod_{k=1}^{n} \binom{m_k}{l_k} [1 - R_k(t)]^{m_k - l_k} [R_k(t)]^{l_k}, \quad (4.1)$$

where $R_i(t)$ be the i^{th} component reliability.

Also, using equation (2.7), it can be expressed in closed form as

$$RII_i = \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m_2} \dots \sum_{l_n=0}^{m_n} \left[\frac{\{\Phi_i(l_1, l_2, \dots, l_n) - R_i(t)\Phi(l_1, l_2, \dots, l_n)\}}{R_i(t)(1 - R_i(t))} \right] \prod_{k=1}^{n} \binom{m_k}{l_k} [1 - R_k(t)]^{m_k - l_k} [R_k(t)]^{l_k}. \quad (4.2)$$

Here, we examine the RIIs within the contexts of series, parallel, and redundant configurations, as follows:

Case (1): If the system has a series configuration and the i^{th} component is not perfectly reliable, i.e., $R_i(t) \neq 1$, then the restricted survival signature $\overline{\Phi}_i(l_1, l_2, \dots, l_n) = 0$. Thus, from equation (4.1), the RII can be obtained as

$$RII_i = \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m_2} \dots \sum_{l_n=0}^{m_n} \left(\frac{\Phi_i(l_1, l_2, \dots, l_n)}{R_i(t)} \right) \prod_{k=1}^{n} \binom{m_k}{l_k} [1 - R_k(t)]^{m_k - l_k} [R_k(t)]^{l_k}.$$

Case (2): If the system has a parallel configuration and the i^{th} component is functioning but not perfectly reliable, then the survival signature becomes unity, thus from equation (4.2), the RII of the i^{th} component can be obtained as

$$RII_i = \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m_2} \dots \sum_{l_n=0}^{m_n} \left(\frac{\{\Phi_i(l_1, l_2, \dots, l_n) - R_i(t)\}}{R_i(t)(1 - R_i(t))} \right) \prod_{k=1}^{n} \binom{m_k}{l_k} [1 - R_k(t)]^{m_k - l_k} [R_k(t)]^{l_k}.$$

Case (3): Assume that a coherent system consists of m independent components of k different types and that the subsystem P_k contains one active component of type k and $r_k - 1$ cold standby components. Then, RII of the i^{th} subsystem (previously it was the i^{th} component of type r in the case without a standby configuration), $i \in \{1,2, \dots, m\}$, is obtained as

$$RII_i^c = \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m_2} \dots \sum_{l_n=0}^{m_n} \frac{\{\Phi_i(l_1, l_2, \dots, l_n) - R_i^C(t)\Phi(l_1, l_2, \dots, l_n)\}}{R_i^C(t)(1 - R_i^C(t))} \prod_{k=1}^{n} \binom{m_k}{l_k} \left[1 - \left(R_k(t) + \sum_{j=1}^{r_k - 1} \int_0^t R_{k,j+1}(t-x)\, f_k^{(j)}(x)\ dx \right) \right]^{m_k - l_k} \left[\left(R_k(t) + \sum_{j=1}^{r_k - 1} \int_0^t R_{k,j+1}(t-x)\, f_k^{(j)}(x)\ dx \right) \right]^{l_k}, \tag{4.3}$$

where $R_i^C(t)$ be the survival function of the subsystem P_r.

Theorem 4.1: Consider a coherent system consisting of subsystems $P_{k,}$ each containing one active component of type k and $r_k - 1$ cold standby components. Assume that all the components in the subsystem P_k follow an exponential distribution with a failure rate λ_k. Let $RII_{i_1}^c$ and $RII_{i_2}^c$ be the RIIs of the i_1^{th} and i_2^{th} subsystem P_{k_1} and P_{k_2}, respectively. Further assume that the restricted survival signatures of i_1^{th} and i_2^{th} components are identical. Consider $R_{k_1}^C(t)$ and $R_{k_2}^C(t)$ be the reliabilities of the subsystems P_{k_1} and P_{k_2}, consisting of r_{k_1} and r_{k_2} components with failure rates $\lambda_{k_1}, \lambda_{k_2}$ respectively. Then:

(i) If $r_{k_1} = r_{k_2}$ and $\lambda_{k_1} > \lambda_{k_2}$ then $RII_{i_1}^c > RII_{i_2}^c$.
(ii) If $\lambda_{k_1} = \lambda_{k_2}$ and $r_{k_2} > r_{k_1}$ then, $RII_{i_1}^c > RII_{i_2}^c$.

Proof: Consider the function $\delta_{i_1 i_2}^c = RII_{i_1}^c - RII_{i_2}^c$. We aim to show that $\delta_{i_1 i_2}^c > 0$ for the assumed failure rates $\lambda_{k_1}, \lambda_{k_2}$. The difference of RIIs for the i_1^{th} and i_2^{th} subsystems is

$$\delta_{i_1 i_2}^c = \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m_2} \dots \sum_{l_n=0}^{m_n} \left\{ \frac{\Phi_{i_1}(l_1, l_2, \dots, l_n)}{R_{k_1}^C(t)} - \frac{\bar{\Phi}_{i_1}(l_1, l_2, \dots, l_n)}{1 - R_{k_1}^C(t)} - \frac{\Phi_{i_2}(l_1, l_2, \dots, l_n)}{R_{k_2}^C(t)} + \frac{\bar{\Phi}_{i_2}(l_1, l_2, \dots, l_n)}{1 - R_{k_2}^C(t)} \right\} \prod_{k=1}^{n} \binom{m_k}{l_k} \left[1 - \left(R_k^C(t) \right) \right]^{m_k - l_k} \left[\left(R_k^C(t) \right) \right]^{l_k},$$

where $R_k^C(t)$ denotes the subsystem reliability of P_k. Given that the $\Phi_{i_1}(l_1, l_2, \dots, l_n)$ is identical to $\Phi_{i_2}(l_1, l_2, \dots, l_n)$, therefore, $\delta_{i_1 i_2}^c$ becomes

$$\delta^c_{i_1 i_2} = \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\dots\sum_{l_n=0}^{m_n}\left(R^C_{k_2}(t) - R^C_{k_1}(t)\right)\left\{\frac{\Phi_{i_1}(l_1,l_2,\dots,l_K)}{R^C_{k_1}(t)\,R^C_{k_2}(t)} + \frac{\overline{\Phi}_{i_1}(l_1,l_2,\dots,l_K)}{\left(1-R^C_{k_1}(t)\right)\,\left(1-R^C_{k_2}(t)\right)}\right\}\prod_{k=1}^{n}\binom{m_k}{l_k}\Big[1 - \left(R^C_k(t)\right)\Big]^{m_k-l_k}\Big[\left(R^C_k(t)\right)\Big]^{l_k}$$

i.e.,

$$\delta^c_{i_1 i_2} = \left(R^C_{k_2}(t) - R^C_{k_1}(t)\right)h(t).$$

Here, $h(t) > 0$, therefore $\delta^c_{i_1 i_2} > 0$ iff $R^C_{k_2}(t) > R^C_{k_1}(t)$.

(4.4)

Also, for the subsystem P_k having $r_k - 1$ exponentially distributed standby components with a common hazard rate λ_k, the reliability $R^C_k(t)$, is

$$R^C_k(t) = \sum_{j=0}^{r_k-1}\frac{(\lambda_k t)^j}{j!}e^{-\lambda_k t} \quad .$$

4.5)

Here $R^C_k(t)$ follows the Erlang distribution, we have considered two cases:

Case (i): Taking the derivative of equation (4.5) with respect to λ_k, we have

:

$$\frac{dR^C_k(t)}{d\lambda_k} = \frac{d}{d\lambda_k}\left(\sum_{j=0}^{r_k-1}\frac{(\lambda_k t)^j}{j!}e^{-\lambda_k t}\right)$$

$$\Rightarrow \frac{dR^C_k(t)}{d\lambda_k} = te^{-\lambda_k t}\left(-\sum_{j=0}^{r_k-1}\frac{(\lambda_k t)^j}{j!} + \sum_{j=0}^{r_k-2}\frac{(\lambda_k t)^j}{(j)!}\right)$$

$$i.e., \qquad \frac{dR^C_k(t)}{d\lambda_k} = te^{-\lambda_k t}\left(-\frac{(\lambda_k t)^{r_k-1}}{(r_k-1)!}\right) < 0\,.$$

Hence, $\frac{dR^C_k(t)}{d\lambda_k} < 0$, which implies that the reliability function $R^C_k(t)$ is strictly decreasing w.r.t. λ_k. Therefore, for $r_{k_1} = r_{k_2}$ and $\lambda_{k_1} > \lambda_{k_2}$, it follows that the $R^C_{k_1}(t) < R^C_{k_2}(t)$. From equation (4.4), the difference function satisfies $\delta^c_{i_1 i_2} > 0$, i.e., $RII^c_{i_1} > RII^c_{i_2}$.

Case (ii): Let $r_{k_1} < r_{k_2}$, and $\Delta(t) = R^C_{k_2}(t) - R^C_{k_1}(t)$, therefore, from equation (4.5)

$$\Delta(t) = \sum_{j=0}^{r_{k_2}-1}\frac{\left(\lambda_{k_2} t\right)^j}{j!}e^{-\lambda_{k_2} t} - \sum_{j=0}^{r_{k_1}-1}\frac{\left(\lambda_{k_1} t\right)^j}{j!}e^{-\lambda_{k_1} t}$$

$$\Delta(t) \quad = \sum_{j=0}^{r_{k_1}-1} \frac{\left(\lambda_{k_2} t\right)^j}{j!} e^{-\lambda_{k_2} t} + \sum_{j=r_{k_1}}^{r_{k_2}-1} \frac{\left(\lambda_{k_2} t\right)^j}{j!} e^{-\lambda_{k_2} t} - \sum_{j=0}^{r_{k_1}-1} \frac{\left(\lambda_{k_1} t\right)^j}{j!} e^{-\lambda_{k_1} t}$$

Given that, $\lambda_{k_1} = \lambda_{k_2} = \lambda_k$, so

$$\Delta(t) = e^{-\lambda_k t} \sum_{j=r_{k_1}}^{r_{k_2}-1} \frac{(\lambda_k t)^j}{j!} > 0,$$

This implies that the $R_{k_2}^C(t) > R_{k_1}^C(t)$. So, it is concluded that for $\lambda_{k_1} = \lambda_{k_2}$, and $r_{k_1} < r_{k_2}$ the difference function $\delta_{i_1 i_2}^c > 0$ i.e., $RII_{i_1}^c > RII_{i_2}^c$. This completes the proof.

Case (4): Assume the subsystem P_k contains one active component of type k and $r_k - 1$ degrading cold standby components. Then, RII of the i^{th} subsystem (previously it was the i^{th} component of type r in the case without standby configuration), $i \in \{1,2, \dots, m\}$, is

$$RII_i^d = \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m_2} \dots \sum_{l_n=0}^{m_n} \frac{\{\Phi_i(l_1, l_2, \dots, l_n) - R_i^D(t)\Phi(l_1, l_2, \dots, l_n)\}}{R_i^D(t)(1 - R_i^D(t))} \prod_{k=1}^{n} \binom{m_k}{l_k} \Bigg[1 - \left(R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t \tilde{R}_{k,j+1}(x)\, R_{k,j+1}(t-x)\, \tilde{f}_k^{(j)}(x)\, dx \right) \Bigg]^{m_k - l_k} \Bigg[\Bigg(R_k(t) + \sum_{j=1}^{r_k-1} \int_0^t \tilde{R}_{k,j+1}(x)\, R_{k,j+1}(t-x)\, \tilde{f}_k^{(j)}(x)\, dx \Bigg) \Bigg]^{l_k}. \tag{4.6}$$

where $R_i^D(t)$ is the reliability of the subsystem P_r.

5. Numerical Example

A six-component bridge system (Figure 1) is analyzed to illustrate the derived results. It consists of six components of two types and was previously examined in [12] to evaluate reliability and the RII. The system is extended by introducing cold standby components, both with and without degradation, to enhance its reliability (Figure 2). Each component of type k is transformed into a subsystem P_k comprising one active and several INID cold standby components, and the resulting changes in system reliability and subsystems' criticality are subsequently examined.

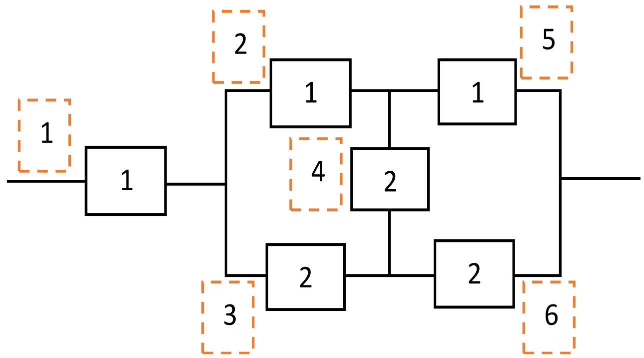

Figure 1. Complex bridge system containing two types of components.

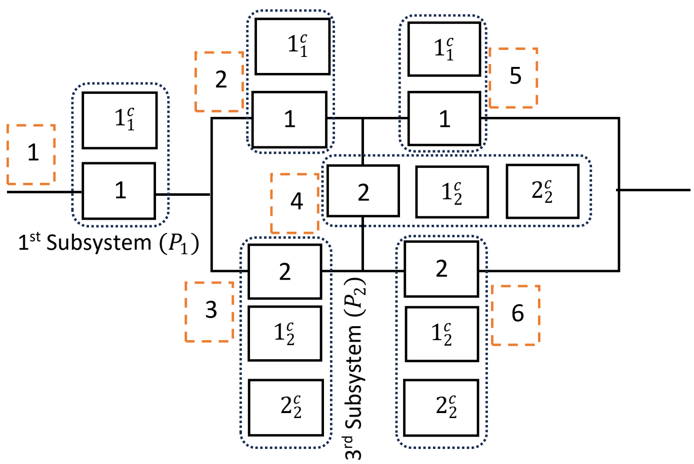

Figure 2. Complex bridge system with cold standby components.

Table 1 summarizes the characteristics of the components, assuming that their lifetimes follow an exponential distribution with distinct failure rates λ. Table 2 presents the survival signature $\Phi(l_1, l_2)$ and the restricted survival signature $\Phi_i(l_1, l_2)$ for l_1 and l_2. Further, the reliability of the considered systems can be evaluated using the data presented in Tables 1 and 2.

Table 1 Components failure rates and their reliability.

Component type	**Failure rate (λ)**	**Component Reliability**
1	0.8	$e^{-(0.8)t}$
1_1^c	2.5	$e^{-(2.5)t}$
2	1.6	$e^{-(1.6)t}$
1_2^c	2.5	$e^{-(2.5)t}$
2_2^c	3.0	$e^{-(3.0)t}$

Table 2. Survival signature and Restricted survival signature.

l_1	l_2	$\Phi(l_1, l_2)$	$\Phi_i(l_1, l_2)$					
			$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
1	2	1/9	1/9	0	1/9	0	0	1/9
1	3	1/3	1/3	0	1/3	1/3	0	1/3
2	2	4/9	4/9	2/9	2/9	2/9	2/9	2/9
2	3	2/3	2/3	2/3	2/3	2/3	2/3	2/3
3	0	1	1	1	1	0	1	0
3	1	1	1	1	1	1	1	1
3	2	1	1	1	1	1	1	1
3	3	1	1	1	1	1	1	1

Using equation (3.4), the reliability of the subsystem P_1 is

$$R_1^C(t) = R_1(t) + \int_0^t R_{1,2}(t-x)\, f_1^{(1)}(x)\ dx$$

where $f_1^{(1)}(x)$ represents the pdf of the first failure arrival for the subsystem P_1. $R_1(t)$ and $R_{1,2}(t)$ denote the reliabilities of the active component and the cold standby component added to the active component in the subsystem P_1, respectively. Since $f_1^{(1)}(x) = f_1(x)$, thus

$$R_1^C(t) = R_1(t) + \int_0^t R_{1,2}(t-x)\, f_1(x)\, dx$$

Also, the reliability of the subsystem P_2 can be found using equation (3.4) as

$$R_2^C(t) = R_2(t) + \sum_{j=1}^{2} \int_0^t R_{2,j+1}(t-x)\, f_2^{(j)}(x)\ dx \tag{5.1}$$

where $f_2^{(j)}(x)$ represents the pdf of the j^{th} failure arrival for the subsystem P_2 i.e, the pdf of the sum of the first j components' failure times. The functions $R_2(t)$ and $R_{2,j+1}(t), j \in \{1, 2\}$ denote the reliabilites of the type 2 active component and cold standby components added to the active component of type 2 in subsystem P_2, respectively. Assuming the components are exponentially distributed, then Case (i): if $j = 1$, then $f_2^{(1)}(x) = f_2(x)$. Case (ii): if $j = 2$, then $f_2^{(2)}(x)$ is pdf of the sum of the lifetimes T_1(lifetime of active component 2) and T_2 (lifetime of the standby component 1_2^c). i.e., $f_2^{(2)}(x) = \int_0^t f_{T_2}(x-u)\, f_{T_1}(u)\, du$.

Given that the lifetimes T_1 and T_2 have failure rates λ_1 and λ_2 respectively, therefore, $f_2^{(2)}(x)$ can be evaluated as

$$f_2^{(2)}(x) = \frac{\lambda_1 \lambda_2 \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)}{\lambda_2 - \lambda_1} \tag{5.2}$$

Using Table 1 and the equation (5.2), the reliability of the subsystem P_2 can be evaluated easily. If the degradation rate of the cold standby components follows a Weibull distribution with parameters $(\alpha_k{}', \beta)$ and the reliability of the degrading component is $R_k(t) = 1 - e^{-\left(\frac{1}{\alpha_k{}' t}\right)^{\beta}}$ as shown in Table 3, then the system reliability can be evaluated using Equation (3.12).

Figure 3 illustrates the system reliability comparison among the systems with no-standby, cold standby, and degrading cold standby components. The no-standby configuration demonstrates a rapid decline in reliability, reflecting poor survivability over time. The inclusion of cold standby components significantly enhances system reliability throughout the mission duration, although degradation during standby diminishes this improvement. Nevertheless, a clear advantage over the no-standby case is maintained. These results validate the efficacy of cold standby redundancy in improving system reliability, even under degrading conditions.

Table 3:Scale and Shape Parameters for Degrading Cold Standby Components

Component	Scale Parameter $\alpha_k{}'$	Shape Parameter β	Degrading Components reliability
1_1^c	1.5	2	$1-e^{-\left(\frac{1}{1.5\,t}\right)^2}$
1_2^c	1.5	2	$1-e^{-\left(\frac{1}{1.5\,t}\right)^2}$
2_2^c	2.0	2	$1-e^{-\left(\frac{1}{2.0\,t}\right)^2}$

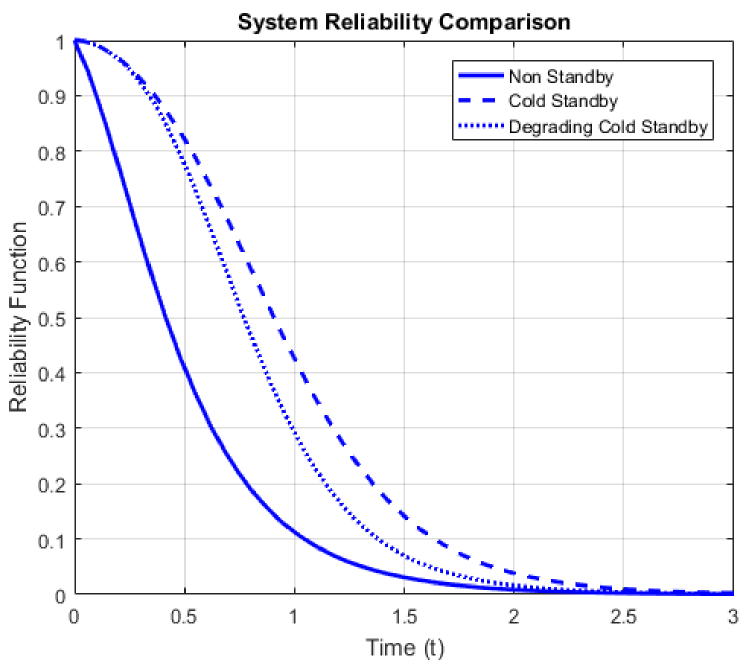

Figure 3: Reliability function of the systems shown in Figures 1 and 2

Using equation (4.4), the RII of the i^{th} component (Figure 1) having reliability $R_i(t)$ can be obtained as

$$RII = \sum_{l_1=0}^{3}\sum_{l_2=0}^{3}\frac{\{\Phi_i(l_1,l_2)-R_i(t)\Phi(l_1,l_2)\}}{R_i(t)\,(1-R_i(t))}\prod_{k=1}^{2}\binom{3}{l_k}[1-R_k(t)]^{n_k-l_k}[R_k(t)]^{l_k}$$

The RII of each component is determined using the data in Tables 1 and 2. Figures 4 and 5 show the RII of components and their corresponding subsystems after incorporating cold standby and degrading cold standby redundancies. Specifically, Figure 4 presents the RII of components 1 and 4 along with the RII of their respective subsystems, while Figure 5 illustrates the RII of components 2 and 3 and their corresponding subsystems under different redundancy strategies. A comparison with [12] shows that the RII of component 4 in the no-standby case matches that obtained by Feng, thereby validating the correctness of the derived RII results. The RII values of the component pairs {2,5} and {3,6} are observed to be identical. For the cold standby and degrading cold standby configurations shown in Figure 2, the RII of the i^{th} subsystem (corresponding to the i^{th} component in Figure 1) is computed using Equations (4.3) and (4.6) along with the data in Tables 1, 2, and 3.

Figures 4 and 5 show that, in the no-standby configuration, component 1 is the most critical, as its RII remains higher than that of all other components over the complete time duration. After the introduction of redundancy, each component is transformed into a subsystem. Under both cold standby and degrading cold standby configurations, subsystem 1 continues to exhibit the highest RII among all subsystems. Compared to the no-standby case, the cold standby strategy increases the RII of subsystem 1, indicating that the system is more sensitive to its operational state. When degradation during standby is considered, the RII of subsystem 1 decreases slightly relative to the ideal cold standby case but remains higher than that of the component in original no-standby configuration for a longer duration. In contrast, component 4 has the lowest RII in the original system, and its corresponding subsystem 4 remains the least critical under redundancy, with its RII declining rapidly after ($t = 1$).

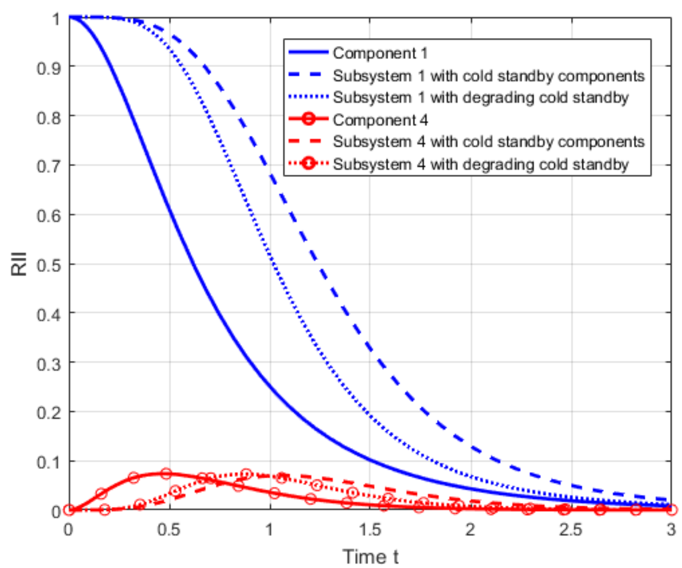

Figure. 4: RII of the 1st and 4th component / subsystem

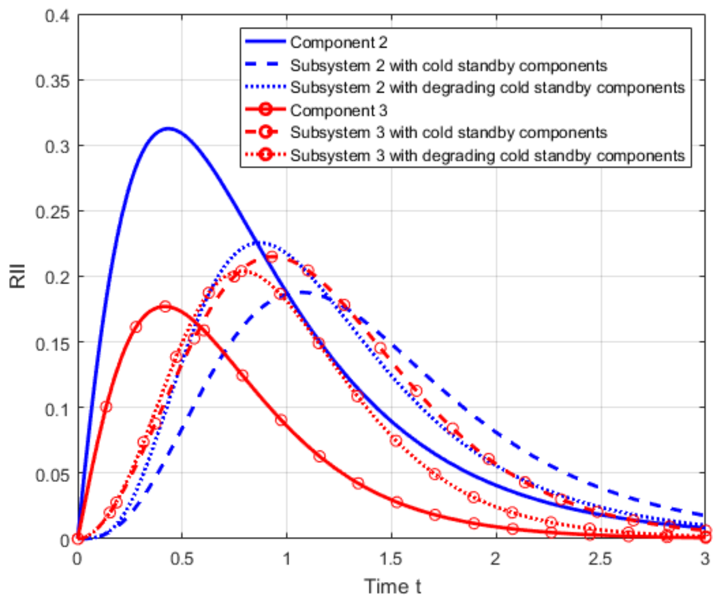

Figure 5.: RII of the 2nd and 3rd component / subsystem

In the absence of standby, Figure 5 shows that component 2 is consistently more critical than component 3 over the entire time frame, and this ranking remains time-invariant. The importance ranking of subsystems is time-dependent after the introduction of redundancy. Under cold standby conditions, the RII values of subsystems 2 and 3 coincide at ($t = 1.3$); for ($t < 1.3$), subsystem 3 has a higher RII than subsystem 2, whereas for ($t > 1.3$), subsystem 2 becomes more critical than subsystem 3. A similar behavior is observed for the degraded cold standby configuration, where the RII values of subsystems 2 and 3 coincide at ($t = 0.7$), after which the criticality shifts from subsystem 3 to subsystem 2. The analysis shows that, despite fixed component importance rankings in a no-standby system, adding redundancy induces time-dependent changes in RII and alters criticality over time.

6. Conclusion

The study demonstrates that a survival signature approach offers a systematic reliability analysis of coherent systems, identifying critical components and their time-dependent influence on performance. This emphasizes the role of component criticality in enhancing system reliability. To address this, the system is augmented with cold standby components, both with and without degradation effects, to improve performance and mitigate the impact of critical components. A closed-form recursive expression for subsystem reliability is derived by modeling each component as a subsystem consisting of one active and many INID cold standby components. Additionally, the RII expression based on the restricted survival signature is formulated, and a theorem for RII is established under the assumption of exponentially distributed cold standby components.

The numerical example demonstrated that cold standby redundancy significantly improves system reliability, even in the presence of degradation. The RII analysis further reveals that redundancy redistributes criticality rather than uniformly reducing it. Whereas subsystems 2 and 3 show time-dependent shifts in importance after redundancy implementation, highlighting the configuration and time-dependent behavior of the subsystem importance. This framework provides useful guidance for the allocation of strategic redundancies. Future research may extend the proposed framework to repairable systems and dynamically evolving system configurations. Additionally, the model can be further developed to incorporate more complex operational environments and maintenance strategies, enabling a broader applicability of the approach in practical reliability analysis.

Reference

1. S. V Amari, H. Pham, and R. B. Misra, “Reliability Characteristics of $ k $-out-of-$ n $ Warm Standby Systems,” *IEEE Trans. Reliab.*, vol. 61, no. 4, pp. 1007–1018, 2012.
2. L. J. M. Aslett, F. P. A. Coolen, and S. P. Wilson, “Bayesian Inference for Reliability of Systems and Networks Using the Survival Signature,” *Risk Anal.*, vol. 35, no. 9, pp. 1640–1651, Sep. 2015, doi: https://doi.org/10.1111/risa.12228.
3. R. E. Barlow and F. Proschan, “Importance of system components and fault tree events,” *Stoch. Process. their Appl.*, vol. 3, no. 2, pp. 153–173, 1975, doi: 10.1016/0304-4149(75)90013-7.
4. Z. W. Birnbaum, “On the importance of different components in a multicomponent system,” Washington Univ Seattle Lab of Statistical Research, 1968. doi: 10.21236/ad0670563.
5. M. Chang, X. Huang, F. P. Coolen, and T. Coolen-Maturi, “New reliability model for complex systems based on stochastic processes and survival signature,” *Eur. J. Oper. Res.*, vol. 309, no. 3, pp. 1349–1364, 2023, doi: 10.1016/j.ejor.2023.02.027.
6. Y. Chen and H. Feng, “Maintenance strategy of multicomponent system based on structure updating and group importance measure,” *Commun. Stat. - Theory Methods*, vol. 51, no. 9, pp. 2919–2935, Mar. 2022, doi: 10.1080/03610926.2020.1784434.
7. D. W. Coit, “Cold-standby redundancy optimization for nonrepairable systems,” *IIE Trans. (Institute Ind. Eng.*, vol. 33, no. 6, pp. 471–478, 2001, doi: 10.1023/A:1007689912305.
8. F. P. A. Coolen and T. Coolen-Maturi, “Generalizing the signature to systems with multiple types of components,” in *Advances in Intelligent and Soft Computing*, vol. 170 AISC, Springer, 2012, pp. 115–130. doi: 10.1007/978-3-642-30662-4-8.
9. S. Eryilmaz, F. P. A. Coolen, and T. Coolen-Maturi, “Marginal and joint reliability importance based on survival signature,” *Reliab. Eng. Syst. Saf.*, vol. 172, pp. 118–128, 2018.
10. S. Eryilmaz and A. Tuncel, “Generalizing the survival signature to unrepairable homogeneous multi-state systems,” *Nav. Res. Logist.*, vol. 63, no. 8, pp. 593–599, 2016.
11. J. Fan, F. Han, and H. Liu, “Challenges of Big Data analysis,” *Natl. Sci. Rev.*, vol. 1, no. 2, pp. 293–314, Jun. 2014, doi: 10.1093/nsr/nwt032.
12. G. Feng, E. Patelli, M. Beer, and F. P. A. Coolen, “Imprecise system reliability and component importance based on survival signature,” *Reliab. Eng. Syst. Saf.*, vol. 150, pp. 116–125, 2016, doi: 10.1016/j.ress.2016.01.019.
13. C. Franko, M. Ozkut, and C. Kan, “Reliability of coherent systems with a single cold standby component,” *J. Comput. Appl. Math.*, vol. 281, pp. 230–238, 2015, doi: 10.1016/j.cam.2014.12.029.
14. Hindolo George-Williams, Geng Feng, Frank P A Coolen, Michael Beer, and Edoardo Patelli, “Extending the survival signature paradigm to complex systems with non-repairable dependent failures,” *Proc. Inst. Mech. Eng. Part O J. Risk Reliab.*, vol. 233, no. 4, pp. 505–519, Aug. 2019, doi: 10.1177/1748006X18808085.
15. C. M. Grinstead and J. L. Snell, *Introduction to Probability*, Revised. USA: American Mathematical Society, 1997. [Online]. Available: https://books.google.co.in/books?id=14oq4uWGCkwC
16. J. Guo, T. Guo, J. Su, J. Li, and X. Liu, “Reliability Estimation of Stress–Strength Model for Multi-Component System Based on Lomax Distribution Using the Survival Signature,” *Symmetry (Basel).*, vol. 17, no. 4, 2025, doi: 10.3390/sym17040614.
17. A. D. Hesampour, K. Ziarati, and S. Zarezadeh, “A novel evolutionary strategy optimization algorithm for reliability redundancy allocation problem with heterogeneous components,” *Swarm and Evolutionary Computation*, vol. 90. 2024. doi: 10.1016/j.swevo.2024.101695.
18. X. Huang, L. J. M. Aslett, and F. P. A. Coolen, “Reliability analysis of general phased mission systems with a new survival signature,” *Reliab. Eng. Syst. Saf.*, vol. 189, pp. 416–422, Sep. 2019, doi: 10.1016/j.ress.2019.04.019.
19. W. Kuo and V. R. Prasad, “An annotated overview of system-reliability optimization,” *IEEE Trans. Reliab.*, vol. 49, no. 2, pp. 176–187, 2000, doi: 10.1109/24.877336.
20. W. Kuo and X. Zhu, *Importance measures in reliability, risk, and optimization: principles and applications*. John Wiley & Sons, 2012. doi: 10.1002/9781118314593.
21. Y. Li, C. Zhu, and Z. Wang, “Reliability analysis and imprecise component importance measure of redundant systems of OWTs based on component swapping,” *Appl. Sci.*, vol. 10, no. 4, 2020, doi: 10.3390/app10041432.
22. C. J. Lu and W. O. Meeker, “Using Degradation Measures to Estimate a Time-to-Failure Distribution,” *Technometrics*, vol. 35, no. 2, pp. 161–174, May 1993, doi: 10.1080/00401706.1993.10485038.
23. E. K. Mutar and Z. A. H. Hassan, “Survival Signature-Based Structural Importance Analysis of Multistate System with Binary-State Components,” *ASCE-ASME J. Risk Uncertain. Eng. Syst. Part B Mech. Eng.*, vol. 11, no. 2, 2025, doi: 10.1115/1.4067826.
24. E. K. Mutar, “Reliability Importance Analysis of a Multi- State System with Binary Components Using Survival Signature,” *J. Aerosp. Technol. Manag.*, vol. 17, pp. 1–24, 2025, doi: 10.1590/jatm.v17.1378.
25. J. Navarro and M. Asadi, “Coherent systems with components subject to additive or multiplicative degradation models,” *Proc. Inst. Mech. Eng. Part O J. Risk Reliab.*, vol. 239, no. 5, pp. 1086–1101, 2025, doi: 10.1177/1748006X241298012.

26. V. Rathod, O. P. Yadav, A. Rathore, and R. Jain, “Reliability-Based Design Optimization Considering Probabilistic Degradation Behavior,” *Qual. Reliab. Eng. Int.*, vol. 28, Dec. 2012, doi: 10.1002/qre.1281.
27. M. Rausand and A. Hoyland, *System reliability theory: models, statistical methods, and applications*, vol. 396. John Wiley & Sons, 2003.
28. S. Reed, “An efficient algorithm for exact computation of system and survival signatures using binary decision diagrams,” *Reliab. Eng. Syst. Saf.*, vol. 165, pp. 257–267, 2017, doi: https://doi.org/10.1016/j.ress.2017.03.036.
29. A. Roy and N. Gupta, “A study on utilization of a cold standby component to enhance the mean residual life function of a coherent system,” *Commun. Stat. - Theory Methods*, vol. 53, no. 19, pp. 6977–6996, 2024, doi: 10.1080/03610926.2023.2255323.
30. F. J. Samaniego, “On Closure of the IFR Class Under Formation of Coherent Systems,” *IEEE Trans. Reliab.*, vol. R-34, no. 1, pp. 69–72, 1985, doi: 10.1109/TR.1985.5221935.
31. Y. Shi, J. Behrensdorf, J. Zhou, Y. Hu, M. Broggi, and M. Beer, “Network reliability analysis through survival signature and machine learning techniques,” *Reliab. Eng. Syst. Saf.*, vol. 242, p. 109806, 2024, doi: https://doi.org/10.1016/j.ress.2023.109806.
32. P. Wang and D. Coit, *Reliability and Degradation Modeling with Random or Uncertain Failure Threshold*. 2007. doi: 10.1109/RAMS.2007.328107.
33. W. Wang, J. Xiong, and M. Xie, “Cold-standby redundancy allocation problem with degrading components,” *Int. J. Gen. Syst.*, vol. 44, no. 7–8, pp. 876–888, 2015, doi: 10.1080/03081079.2015.1028541.
34. S. Wu and F. P. A. Coolen, “A cost-based importance measure for system components: An extension of the Birnbaum importance,” *Eur. J. Oper. Res.*, vol. 225, no. 1, pp. 189–195, 2013, doi: https://doi.org/10.1016/j.ejor.2012.09.034.
35. X. Wu, T. Huang, and J. Liu, “A general multivariate gamma process with moment matching estimation for degradation modeling,” *Reliab. Eng. Syst. Saf.*, vol. 270, 2026, doi: 10.1016/j.ress.2025.112151.
36. F. Yalcin and C. Franko, “Computing Birnbaum and Barlow-Proschan importance measures using system signature,” *Commun. Stat. - Theory Methods*, vol. 51, no. 23, pp. 8446–8457, 2022, doi: 10.1080/03610926.2021.1986536.
37. A. Zhang, S. Wu, D. Fan, M. Xie, B. Cai, and Y. Liu, “Adaptive testing policy for multi-state systems with application to the degrading final elements in safety-instrumented systems,” *Reliab. Eng. Syst. Saf.*, vol. 221, p. 108360, 2022.
38. Y. Zheng, Y. Zhang, and J. Lin, “System reliability analysis for independent and nonidentical components based on survival signature,” *Probabilistic Eng. Mech.*, vol. 73, p. 103466, 2023, doi: https://doi.org/10.1016/j.probengmech.2023.103466.
39. E. Zio and L. Podofillini, “Accounting for components interactions in the differential importance measure,” *Reliab. Eng. Syst. Saf.*, vol. 91, no. 10, pp. 1163–1174, 2006, doi: https://doi.org/10.1016/j.ress.2005.11.044.