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A Group Decision-Making Model with Comparative Linguistic Expression Based on Hedge Algebra

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***Abstract*: The Group Decision Making (GDM) is a problem that gives a synthetic choice from various assessments of some evaluators. It is the complexity of the problem that is the difference among the qualitative evaluations of evaluators. Apparently, this depends on many factors as well as the credibility of each evaluator. The final synthetic decision to make must ensure accuracy and contain all opinions from all evaluators. In this paper, we have proposed a new computational algorithm for assessments using the comparative linguistic expressions that use hedge algebras. Under this approach, we can easily calculate the semantic value of language terms. The proposed algorithm also ensures the logical and simple rigor of execution. We have applied the proposed algorithm on a specific application that is taking the group decision-making from the reviewers to choose the highest rated article to award the author. The calculation results showed the correctness and efficiency of the algorithm.**

***Keywords*:** group decision-making, comparatively linguistic expression, hedge algebra, semantic value of the linguistic.

I. Introduction

In recent years, the group decision problem has been widely researched for its application in many fields such as economics, society, science. The group decision problem can be expressed as follows: A group of m experts evaluate and evaluates n objects according to k criteria and then, based on evaluations, uses a determining method to select the most suitable object. Problems can become complex for two common reasons. Firstly, a clear assessment (such as scoring) is not always easy due to uncertainty in the expert's comments. Secondly, conventional experts rarely achieve a consensus in the assessment, so compromises must be made to achieve the most consistent results based on a number of determining criteria.

For the first problem, through solving simple problems such as experts' evaluation based the scores on a specific scorecard (for example, from 1 to 100), more research have been done on the change direction of this restriction appropriate with human's thinking. For example, this allows experts to evaluate in terms of natural language (in a given set of elements or more broadly, any familiar natural words can be used). Moreover, in difficult cases, experts can use a set of words to evaluate, for example, these are the "good" or "quite good" words [1], or uses conjugated words to form a new phrase. Tang [2] introduced a linguistic model that manages the linguistic explanation constructed by the logical join ($\wedge$, $\vee$, $\rightarrow$, ...) and the fuzzy relation, using the comparative form to evaluate each pair of objects, for example, "A is much better than B" and so on. This development is indispensable in practice, but obviously makes the problem more complex as it is associated with computation with the computing with word (CWW).

In additions, there are often solutions to interact directly with the experts, changing the assessment to the final, after some rounds of compromise, the highest consensus can be based on some given consent measurement [3], [4].

In [5], the authors have introduced a new concept of comparison, which is a range comparison. Based on that, we proposed algorithm of Group Decision Making with testimonials by comparing each pair. However, the comparison with the existing algorithms has not been studied. When solving a decision problem with linguistic information, the steps that are commonly used are,

1) Choose the set of linguistic words to be used for the evaluation and the method for processing this set
2) Select the method for aggregating the comments
3) Choose the best option by an implementation phase and exploitation phase.

In this paper, we focus on the first issue of choosing the linguistic term sets (LTS) for the evaluation and method of processing this term set. There are a number of key methods to choose,

- Select the given LTS, this way is simple but limited experts' opinion.
- Allows selection of either the given LTS or the LTS-generated expression (usually in the grammatical form of a context-free language) [4].
- Allows selection of a hesitant fuzzy linguistic term set with its possible distribution. For example, evaluating a football player in a season is "very good/30%", "good/50%", "average/20%".

The methods are relatively diverse and have obtained good results. However, the problem arises when converting from natural language to digital, so that it can be processed on a computer. Most current methods use the order index of the term in LTS as the basis for the conversion, whether it is direct or indirect. For example, using set 2 (2-tuple) [3], despite the addition of a variable parameter to calculate the deviation from the exact position of the index, is the approximation of the ordinal number. This allows for relatively simple processing and gives good results in cases which the scale is distributed fairly evenly in the specified domain. However, this is not always obvious in reality. For example, on a scale of 10 at a school, the point 1 or 2 is quite far from the rest.
Hedge algebra has been proposed by the group of authors in [14] which is an algebraic approach on semantic domain of linguistic terms, allowing to represent and calculate on problems that are stated by linguistic models.

Studies on hedge algebras are extended and applied, such as automatic classification problem [16], [17], [18]. In [19], the authors also point out its superiority of hedge algebra in real-world representation through fuzzy systems, which is the premise for the application of HA in many different fields.
In this paper, the approach based on hedge algebra is used. This approach partly relieves the above weakness due to the rather strict math structure set on the item's defined domain and the flexibility of determining the range between the scores which are set by the elements of the hedge algebras on this identified domain. Specifically, it is our focuses on the 1st step in the three-step process mentioned above that is our new approach. The remaining steps can now use known methods [4]. In addition, we particularly study the evaluation of comparative language expressions, a more complex form of evaluation which uses only single language terms.

II. Some basic concepts

Decision-making is a frequent process for human activities in many areas. The complexity of decision-making encompasses different perspectives. To achieve this, an effective solution is needed from the knowledge provided by a team of experts. The decision made has to ensure accuracy and contain comments from all members of the group. GDM is often quite complex. This complexity derives from the qualitative evaluation of the experts, but not from uncertain one as well as their different opinions.

There has been a lot of research developed based GDM related issues. One of them is GDM from experts when evaluating subjects from a comparative point of view using natural language words. A model of computation should be constructed from a set of expert opinions given in the form of comparison expressions in natural language to obtain the most common alternative. Through a qualitative point of view using the language, after entering, we can quantify to choose the best option with the highest rating.

A. GDM problem with expert opinion by comparative linguistic expressions

Suppose there are m experts $E = \{e_1, e_2, \dots, e_m\}, (m \geq 2)$ that evaluate n objects or solutions $A = \{a_1, a_2, \dots, a_n\}, (n \geq 2)$ and are expressed by the natural language comparison between an object a_i to a_j [4]. For example, "*a_i is a little more than a_j*", "*a_i a little more than a_j*", "*compared a_i to a_j in very small to moderate range*". Each expert e_k will give a matrix P^k that notes the result of the comparison for $n(n+1)/2$ objects.

$$P^k = \begin{bmatrix} p_{11}^k & p_{12}^k & \dots & p_{1n}^k \\ p_{21}^k & \dots & \dots & p_{2n}^k \\ \dots & \dots & p_{ij}^k & \dots \\ p_{n1}^k & p_{n2}^k & \dots & p_{nn}^k \end{bmatrix}; \; i,j = 1..n; k = 1..m \quad (1)$$

Each evaluation p_{ij}^k represents the degree of appreciation of the "more satisfied" relationship between the alternative a_i versus a_j according to the expert e_k. The problem is to synthesize, or aggregate these ideas to some extension to arrange the given object (s). Therefore, we can choose the best evaluation options.

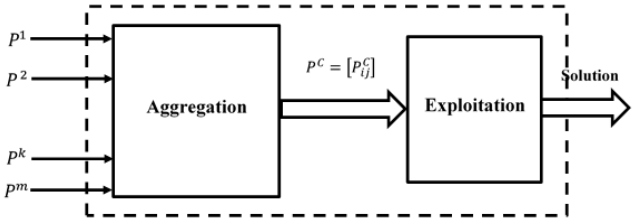

Figure 1. Overview of the group decision model

In Figure 1, the process of collecting, processing, and decision-making options in the group decision problem is shown. In the above schema, it can be seen that it is necessary to perform the following two major phases for GDM:

- **Aggregation phase**: Collect expert's opinions (evaluation matrices P^k; Convert comparison expression in languages to the representation by language intervals for computational models; Incorporate evaluations from experts e_k.
- **Exploitation phase**: Calculate the degree of evaluation for each alternative by the experts e_k; Sort, choose the option with the best alternative.

B. Overview of hedge algebra

Hedge algebra (HA) is a new approach to the set of natural linguistic terms. Unlike the fuzzy set theory, which extends the classic set of concepts, the hedge algebra is derived from the construction of a hedge algebraic structure on the set of elements of a linguistic variable. This algebraic structure is based on the inherent natural linguistic order of the set of linguistic terms. For example, with the linguistic variable "age" we can get a set of terms whose value of that variable is AGE = {"very young", "young", ... "middle aged", "quite old", "old", "very old" ...}, in which in natural semantics, there is always an order "very young" < "young" <…< "middle aged" < "quite old" < "old" < "very old" … If we consider the two words "young" and "old" as 2 generating elements and the highlight words that act on them are "very", "quite" … Then we have the new elements which are "very old", "quite old". This can be considered the result of unary operator between the hedge "very" and the element "old". This unary operator can be repeated several times. For example, if we continue to act, we get "very very old" or "very old". We then have an algebraic structure on the domain of the linguistic variable, defined by the set of 4-tuple) $AX = (X, G, H, \leq)$.

Definition 1. [14]. Hedge Algebra is denoted by the $AX =$

$(X, G, H, \preceq)$ which is the set of 4-tuple, where:

- X the domain of the linguistic variable,
- G is the set of generating elements and constant,
- H is set of hedges which unary operators act the elements of X,
- "$\preceq$" is the semantically induced relation on X.

Suppose that in G there are elements of constants ***0, 1, W*** whose meaning is the smallest element, the largest element, and the neutral element in X. We call each linguistic value $x \in X$ a term in hedge algebra.

It is always assumed that the hedges in H are sequential operators. It means $(\forall h \in H, h: X \to X)$, $(\forall x \in X)\{hx \leq x \text{ or } hx \geq x\}$. The two hedges $h, k \in H$ called reverse if $(\forall x \in X)\{hx \leq x$ if and not if $kx \geq x\}$ and they are called compatible if $(\forall x \in X)\{hx \leq x$ if and not if $kx \leq x\}$.
We can denote $h \geq k$ if h, k are compatible $(\forall x \in T)$ $\{hx \leq kx \leq x$ or $hx \geq kx \geq x\}$.

In addition, the set of H can also be partitioned into two sets H^+ and H^- where the hedges in the set H^+ or H^- are compatible, each element in H^+ is reverse to any elements in H^- and vice versa.

Suppose that in the set of H^+ there is the element V (implicitly $Very$) and in the set of H^- there is the element L (implicitly $Little$) which is the largest element, then the generating element $c \in G$ is positive if $c \leq Vc$ (denoted as c^+) and it is positive if $c \geq Vc$ (called c^-) (or $c \in G$ is positive if $c \geq Lc$ and positive if $c \leq Lc$).

The hedge h is positive (or negative) to one hedge k if $(\forall x \in X)$ $\{hkx \leq kx \leq x \text{ or } hkx \geq kx \geq x\}$ $(or$ $(\forall x \in X)\{kx \leq hkx \leq x \text{ or } kx \geq hkx \geq x\})$.

T is generated from G by the hedges in H. Therefore, each element of T will have the form of representation $x = h_n h_{n-1} \dots h_1 c, c \in G$.

If the set X and H are linear sequential sets, then $AX = (X, G, H, \preceq)$ is the linear HA. Moreover, usually in applications, the value domain of a linguistic variable consists of elements that are generated from two symmetric generating elements (such as "young" vs "old", "far" vs "near", "tall" vs "short". Therefore, in this paper, hedge algebra means that the linear hedge algebra which has two symmetry generating elements, denoted as c^- and c^+. Thus, G is the set which has the order $G = \{0 < c^- < W < c^+ < 1\}$.

When impacting the hedge $h \in H$ on the term x $\in$ X, the element hx is obtained. For every $x \in$X, the notation $H(x)$ is the set of all terms $u \in X$ that are generated from x by applying the hedges in H *and* $u = h_n \dots h_1 x$, where $h_n \dots h_1 \in H$, n is the length of u, denoted as *l(u)*. The set H consists of positive hedges H^+ and the negative one H^-. Positive hedges increase the semantics of a term which it affects, and negative hedges decrease the semantics of the element. In this paper, we limit the study of the two hedges, the corresponding denoted V $\in H^+$ and L $\in H^-$(*V-Very, L-Little*). To represent the positive, the negative of a hedge to a term x, we have the Sign function. If the hedge h is positive to x then Sign (hx) = 1; otherwise, *Sign (hx)* = - 1.

To solve real problems using the theory of fuzzy set, a fuzzy concept must be quantified through membership functions. In HA, linguistic values can also be quantified. The difference is that due to the properties of the HA, this quantification preserves the orderly relations that exist between the categories and allows to retain the semantics of the terms and the result of the integration while performing some transformations. This is almost impossible for the use of membership functions based on Zadeh. The quantification of terms of HA is carried out through the concept of blurring of the linguistic values.

The set of terms derived from x by the action of the hedges will represent the fuzziness of x, denoted by *H(x)*, which denotes the fuzziness of *x*. The size of the set *H(x)* is the measure of the fuzziness of *x*. We can construct the fuzzy measure function *fm* as a mapping from *X* to [0,1], mapping the set *H (x)* to *fm (x)*, satisfying some properties as axioms.

Next, we construct the intervals $\mathfrak{I}(x)$ of length *fm(x)* and range in the segment [0,1] (which is the normalized semantic value of *X*) in the corresponding order with the order of x. Relying on the characters of the HA, for all sets of x_i with a given length k, we have a partition of the segment [0,1] into the fuzzy spaces $\mathfrak{I}(x_i)$. Next, in each fuzzy interval $\mathfrak{I}(x_i)$ we can choose a point that represents both the linguistic values x_i, called the semantic quantifier value -$\upsilon(x_i)$, in a defined formula through a recursive calculation, derived from given parameters of HA.

These are parameters *fm(c-)*, *fm(c+)*, $\mu(h_j)$, $h_j \in H$, where $\mu(h_j)$ is the fuzzy measurement of h_j. Moreover, according to the properties of HA, it is easy to see the semantic quantitative value υ(x) of the element x is the common endpoint of the two fuzzy ranges $\mathfrak{I}(Lx)$ and $\mathfrak{I}(Vx)$. This value divides the fuzzy distance $\mathfrak{I}(x)$ by the ratio $\alpha{:}\beta$ if *Sign(Vx)=1*, or $\beta{:}\alpha$ if *Sign(Vx)=-1*, where α, β are respectively the fuzzy measurement $\mu(L)$ and $\mu(V)$.

The important character of these fuzzy ranges is that they are orderly arranged and with elements of the same length, they form a partition on the specified domain of linguistic variables. Since creating a partition, any element that is the value of a linguistic variable must belong to one of these fuzzy ranges. In addition, if further consideration is made, the fuzzy distance of the term *k* may be further partitioned by fuzzy distances of length *k+1* (with the same root, shown in Figure 2). Therefore, when finishing the constructing of the HA structure, we consider having a complete LTS set. In fact, of course, there is no infinite number of words of this HA, but we are usually limited to word of length *k<4*, then we always have a good approximation to any word that is the value of linguistic variables. In other words, a certain element of any expert uses the attribute of an object, for example, it can always be approximated by an element of HA with a given proximity.

For example, we consider HA $AX = (X, G, H, \leq)$ with $H^- = \{R, L\}$ and $H^+ = \{V, E\}$. A segment of the fuzzy distance system associated with X is shown in Figure 2. We can see $\{\mathfrak{I}(Vy): h \in H\} = \{\mathfrak{I}_3(EVy), \mathfrak{I}_3(VVy), \mathfrak{I}_3(RVy), \mathfrak{I}_3(LVy)\}$ constitutes a partition of level three of the fuzzy ranges $\mathfrak{I}_2(Vy)$ at level 2.

The value of $\upsilon(Vy)$ is the end of the two fuzzy ranges of level 3 $\mathfrak{I}_3(VVy)$ và $\mathfrak{I}_3(RVy)$, and $\upsilon(y)$ is the tip of the fuzzy interval of level 3 $\mathfrak{I}_3(LVy)$ and $\mathfrak{I}_3(EVy)$ and of the two fuzzy ranges of level 2 $\mathfrak{I}_2(Vy)$ and $\mathfrak{I}_2(Ry)$.

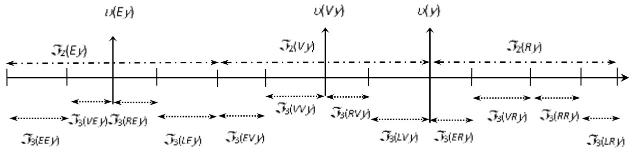

Figure 2. Fuzzy range

According to the above properties, we have a recursive definition of the sign function as follows:

Definition 2. [14]. The function $sgn: X \to \{-1, 0, 1\}$
Where $k, h \in H$, $c \in G$, $u \in X$:

- $sgn(c^+) = +1$ and $sgn(c^-) = -1$
- $\{h \in H^+ | sgn(h) = +1\}$ and $\{h \in H^- | sgn(h) = -1\}$
- $sgn(hc^+) = +sgn(c^+)$ if $hc^+ \geq c^+$ or $sgn(hc^-) = +sgn(c^-)$ if $hc^- \leq c^-$ and $sgn(hc^+) = -sgn(c^+)$ if $hc^+ \leq c^+$ or $sgn(hc^-) = -sgn(c^-)$ if $hc^- \geq c^-$. Or $sgn(hc) = sgn(h)sgn(c)$.
- $sgn(khu) = +sgn(hu)$ if k is positive if h ($sgn(k, h) = +1$) and $sgn(khu) = -sgn(hu)$ if k is positive if h ($sgn(k, h) = -1$).
- $sgn(khu) = 0$ if $khu = hu$.

Proposition 1. [14]: $x \in X$, $x = h_n h_{n-1} \dots h_1 c$, $h_j \in H$, $c \in G$. Then:

$$sgn(x) = sgn(h_n, h_{n-1}) \dots sgn(h_2, h_1) sgn(h_1) sgn(c)$$

$(sgn(hx) = +1) \Rightarrow (hx \geq x)$ and $(sgn(hx) = -1) \Rightarrow (hx \leq x)$

The sign function sgn is used to determine the impact dimension if it increases or decreases the semantic value of a hedge to a linguistic value.

An algebraic structure $AX = (X, G, H, \leq)$ with H is partitioned into H^+ and H^- inverse hedges are called a hedge algebra if it satisfies the following axiomatic:

1. Each hedge is either positive or negative for any other, including itself.
2. If the two concepts u and v are independent, i.e. $u \notin H(v)$ and $v \notin H(u)$, then $(\forall x \in H(u))\ \{x \notin H(v)\}$. Moreover, if u and v cannot be compatible the any $x \in H(u)$ will not be compatible if $y \in H(v)$. ($H(u)$ is the set of values generated by the impact of the hedges H on u).
3. If $u \neq hu$ then $u \notin H(hu)$ and if $h \neq k$ and $hu < ku$ then $h'hu \leq k'ku$, with every hedge h, k, h' and k'. In additions, $hu \neq ku$ where hu and kuare independent.
4. If $u \notin H(v)$ and $u \leq v$ (or $u \geq v$) then $u \leq v$ (or $u \geq hv$) to every hedge h.

In terms of the hedge algebra AX, there are just two generating elements: negative, positive and a neutral element w between two generating elements and $hw = w$, with any $h \in H$. An element v is called the inverse element of the element u if there exists a representation of u represented as $u = h_n h_{n-1} \dots h_1 c$, $w \neq c \notin G$, where $v = h_n h_{n-1} \dots h_1 c'$, with $w \neq c' \notin G$ and $c' \neq c$ (in other words: two generating elements of the hedge algebra are called contradictory if they are represented by the same sequence of hedges but their generating element is different, one is positive and one is negative).

Particularly, the opposite part of w is defined as w. The opposite element of u is denoted by $-u$ with an index if necessary. In general, an element can have many opposing elements.

If each element of X has only one opposing element, then AX is called a symmetric hedge algebra.
The following theorem demonstrates the semantic ordering of the linguistic elements in the hedge algebra.

Theorem 1. [14]. Let the set H^- and H^+ be the linear ordering sets of hedge algebra $AX = (X, G, H, \leq)$. Then we have the following assertions:
For each $x \in X$, $H(x)$ is a linear sequential set.
If X is derived from G by hedges and G is a linear order, X is also a linear order. Furthermore, if $u < v$, and u, v are independent of each other, i.e. $u \notin H(v)$ and $v \notin H(u)$, then $H(u) \leq H(v)$.

Theorem 2. [14]. A hedge algebra AX is symmetric if any u, u is a stopping point if and only if $-u$ is also a stop.
The above theorem demonstrates that the hedge algebra is symmetric, though only based on the natural properties of the concept of language, it has properties which are very important and sufficient to build and develop a logical basis for the approximate reasoning. Obviously, it would be a non-classical logic. Moreover, it can also be seen that G is the symmetrical hedge algebra of AX and satisfies the properties of algebra for 3-valued logic. For that reason it is possible to view each symmetric syllabus as an algebraic base for a logic of linguistic values. The next theorem deals the relationship with the domain [0,1].

Theorem 3. [14]. If the set of H^+ and H^- are linearly arranged, there exists an isomorphic φ from the symmetric hedge algebra $AX = (X, G, H, -, \cup, \cap, \Rightarrow, \leq)$ into the multi-valued logic structure on the $[0, 1]$ such that:

1. Assure order relations
2. $\varphi(u \cup v) = max\{\varphi(u), \varphi(u \cup v)\} = min\{\varphi(u), \varphi(v)\}$.
3. $\varphi(u \Rightarrow v) = max\{1 - \varphi(u), \varphi(v)\}$ and $\varphi(-u) = 1 - \varphi(u)$.

C. Measuring functions on hedge algebra

According to Theorem 3, there exists an isomorphic φ between an extension symmetrical hedge algebras and a multi-valued logical structure based on the domain $[0, 1]$. This allows us to set up a measuring function on the hedge algebra to convert a value of the symmetric extended hedge algebra (the class of hedge algebra interested in this subject) into a real value in the domain $[0, 1]$. To construct the measuring function, we assume that the basis hx can be compared. If they do not match, we consider it synonymous and there is only one representative in the hedge algebra. This assumption turns the hedge algebra into a linear sequential set.

Definition 3. [14]. Measuring function on hedge algebra

Suppose the symmetrical extended hedge algebras $\mathcal{AT} = (T, G, C, H, \leq)$, $f: T \to [0, 1]$ is a measuring function on T if it meets the demand:

1. $\forall u \in T: f(u) \in [0, 1], f(c^+) = 1, f(c^-) = 0$; where: $c^+, c^- \in G$, are positive and negative generating elements.
2. $\forall u, v \in T$, if $u < v$ then $f(u) < f(v)$.

Definition 4. [14]. Reverse function of the measuring function Let a hedge algebra $AX = (X, G, H, \lesssim)$, f is a measuring function on X, $f^{-1}: [0, 1] \to X$ is a reverse function of the measuring function f if it satisfies as follows:
$\forall a \in [0, 1], f^{-1}(a) \in X$ where $|f(f^{-1}(a)) - a| \leq |(f(x) - a|, \forall x \in X$.
Based on the above definitions, we have the following theorem:

Theorem 4. [14]. Let the symmetrical extended hedge algebra $\mathcal{AT} = (T, G, C, H, \leq)$, f is a measuring function on T, f^{-1} is the inverse of the measuring function f, we have:

1. $\forall u \in T, f^{-1}(f(u)) = u$
2. $\forall a, b \in [0, 1]$, if $a \leq b$ then $f^{-1}(a) \leq f^{-1}(b)$

Each symmetrical hedge algebra can define the measuring function and its inverse one because of the coexistence between the hedge algebras and the domain $[0, 1]$. Assuming that the hedges in H are the same, it makes the definition of the function easier.

Definition 5. [14]. Given a hedge algebra $AX = (X, G, H, \lesssim)$. The function $fm: X \to [0, 1]$ called the fuzzy measurement of the elements in X if:

1. $fm(c^-) + fm(c^+) = 1$ and $\sum_{h \in H} fm(hu) = fm(u)$, where $\forall u \in X$ (2)
2. $fm(u) = 0$ where $\forall u$, $H(u) = \{u\}$, $fm(0) = fm(W) = fm(1) = 0$ (3)
3. $\forall u, v \in X, h \in H, \frac{fm(hu)}{fm(u)} = \frac{fm(hv)}{fm(v)}$ (4)

This rate is independent on u, v, but represents for the fuzzy measurement of the hedge h, denoted by $\mu(h)$.

Where, the condition 1 express the completeness of the generating elements and the hedges for the semantic representation of the real domain for the variables. Condition 2 demonstrates the clarity of the terms and conditions 3 can be accepted because we have accepted the assumption that the hedges are context-independent, so when applying a hedge h to elements, relatively affected effects in changing the semantics of those terms are the same.

The properties of the fuzzy measurement of the elements and hedges are shown in the following proposition:

Proposition 2. [14]. Let fm be the function of the fuzzy measurement on X. With $u \in X$, $u = h_n h_{n-1} \ldots h_1 c$, $h_j \in H$, $c \in G$. We have:

1. $fm(hu) = \mu(h) fm(u)$ (5)
2. $\sum_{-q<i<p, i\neq 0} fm(h_i c) = fm(c)$ (6)
3. $\sum_{-q<i<p, i\neq 0} fm(h_i u) = fm(u)$ (7)
4. $fm(u) = fm(h_n h_{n-1} \ldots h_1 c) = \mu(h_n)\mu(h_{n-1}) \ldots \mu(h_1) fm(c)$ (8)
5. $\sum_{i=-1}^{-q} \mu(h_i) = \alpha$ and $\sum_{i=1}^{p} \mu(h_i) = \beta$, where $\alpha, \beta > 0$ and $\alpha + \beta = 1$ (9)

Normally, the semantics of elements are purely qualitative. However, in many applications, it is essential to have the quantitative value of these elements for computation and processing. According to the approach of fuzzy sets, the quantification of fuzzy concepts is done through defuzzification. For the hedge algebras, the quantitative value of the elements is defined based on the semantic order structure of the value domain of the linguistic variables, namely the fuzziness measurement of the elements and hedges.

With a defined set of fuzzy parameters, quantitative semantic values are recursively defined by Semantically Quantifying Mapping (SQM) $\boldsymbol{v}$ as follows:

Definition 6. [14]. Function of Semantically Quantifying Mapping $v: X \to [0, 1]$

1. $v(W) = \theta = fm(c^-)$ (10)
2. $v(c^-) = \theta - \alpha fm(c^-) = \beta fm(c^-)$ (11)
3. $v(c^+) = \theta + \alpha fm(c^+) = 1 - \beta fm(c^+)$ (12)
4. $v(h_j u) = v(u) + sgn(h_j u)\left\{\left[\sum_{i=sgn(j)}^{j} fm(h_i u)\right] - \omega(h_j u) fm(h_j u)\right\}$ (13)

Where:
$\omega(h_j u) = \frac{1}{2}[1 + sgn(h_p, h_j)(\beta - \alpha)]$, $j \in [-q \wedge p] = [-q, p] \backslash \{0\}$

Function of Semantically Quantifying Mapping can be directly mapped from the linguistic value into the semantically quantifying value. Thus, based on SQMs, Computing With Words can be constructed, applied in many mathematical problems expressed in language. Forms of mathematics may include as follows: approximate reasoning based on linguistic rules, fuzzy control; fuzzy association rules; database summary in the form of linguistic rules; clustering problem, fuzzy classification; fuzzy database; regression, fuzzy time series.

HA has the following advantages:

- Always approximate, no need to limit he set of LTS, meet expert needs.
- Converting to more basic numbers. The quantifying semantic value representing the element does not need to be evenly distributed as the order index of the other methods. The semantic quantitative values are always associated with the semantics of the linguistic labels, even after transformations and calculations because they are always within the fuzzy range of the linguistic labels. Therefore, this always ensures the natural order between these labels and makes it different from fuzzy set theory.
- It is relatively simple to handle, especially when we are limited by HA two hedges (Little and Very).

D. Comparatively linguistic expressions and computing methods

First, we need to build the set *X* generated from the previously designed hedge algebra. Select the structure of hedge algebra $AX = (X, G, H, \leq)$ with the following fuzzy sets and fuzzy parameters:

$G = \{neither < low < medium < high < absolute\}$,

$H = \{H^- = \{Little\} \cup H^+ = \{Very\}\}$,
$fm(low) = 0.5, \alpha = \mu(Little) = 0.5$

The set of linguistic elements generated from the hedge algebra can be used for evaluation:

$$X = \{neither < Very\ low < low < Little\ low < medium < Little\ high < high < Very\ high < absolute\} \quad (14)$$

Definition 7. Linguistic expression X_{le}. Propose a set of semantically linguistic terms $X = \{x_1 < x_2 < \cdots < x_g\}$, The structures of comparatively linguistic expression includes:

$$x_i | x_i \in X \quad (15)$$

$$lower\ than\ x_i, greater\ than\ x_i, at\ least\ x_i,$$

$$at\ most\ x_i | x_i \in X \quad (16)$$

$$between\ x_i\ and\ x_j | x_i, x_j \in X, x_i \leq x_j \quad (17)$$

Note: The unary operators "$at\ least$", "$at\ most$" can be understood as the relationship between "$greater\ or\ equal\ to$" and "$lower\ or\ equal\ to$".

Linguistic expressions can be the form of $x_i \in X$ or combined by a form of comparison with $x_i \in X$. Both types of expressions define a range of linguistic values and symbols as the set of X_{le}.

The expert e_1 can describe "better satisfaction" of books with comparative linguistic expressions (15) - (17) and linguistic terms in (14), such as:

$$P^1 = \begin{bmatrix} - & between\ high\ and\ very\ high & very\ high \\ at\ most\ low & - & high \\ at\ most\ low & between\ very\ low\ and\ low & - \end{bmatrix}$$

Definition 8. Domain of linguistic elements R_X. Assuming a set of linguistic terms $= \{x_1 < x_2 < \cdots < x_g\}$, the range of linguistic terms R_x is a continuous subdomain of elements in X.

$$R_X = [x_i < x_{i+1} < \cdots < x_k], x_1 \leq x_i, x_k \leq x_g \quad (18)$$

Definition 9. Function of linguistic conversion E_{R_X}. Propose a set of linguistic terms $X = \{x_1 < x_2 < \cdots < x_g\}$, $le \in X_{le}$ is the comparatively linguistic expressions. The function $E_{R_X}: X_{le} \rightarrow R_X$ is a function of converting each comparatively linguistic expression $le \in X_{le}$ into the ranges of linguistic terms R_X.

$$E_{R_X}(x_i) = [x_i, x_i], x_i \in X \quad (19)$$

$$E_{R_X}(at\ most\ x_i) = [x_j..x_i], j \leq i, length(x_j) = length(x_i) \quad (20)$$

$$E_{R_X}(lower\ than\ x_i) = [x_j..x_i], j < i, length(x_j) = length(x_i) \quad (21)$$

$$E_{R_X}(at\ least\ x_i) = [x_i..x_j], i \leq j, length(x_j) = length(x_i) \quad (22)$$

$$E_{R_X}(greater\ than\ x_i) = [x_i..x_j], i < j, length(x_j) = length(x_i) \quad (23)$$

$$E_{R_T}(between\ x_i\ and\ x_j) = [x_i..x_j], i < j \quad (24)$$

Definition 10. **The semantically linguistic domain of R_X**. The function $v: R_X \rightarrow [R_{X^-}, R_{X^+}]$, define the range of language which R_{X^-} is the semantic value of the element "adjacent left" and R_{X^+} is the semantic value of the element "adjacent right".

$$v(R_X) = [R_{X^-}, R_{X^+}] = [v(R_X(x_l)), v(R_X(x_r))] \quad (25)$$

Thus, applying functions in definitions 9 and 10 can convert comparatively linguistic expressions to their semantic value range. Then, it is possible to perform calculations on this semantic value range such as aggregation, comparison.

III. Algorithm to solve GDM Problem

The algorithm to solve the GDM problem with the experts' evaluation by the expression of comparatively linguistic expression based HA approach is proposed as follows:

```
Algorithm GDM_HA;
  1) Ax = Make(G,H,fuzziness parameters);
  // Make the structureof hedge algebra
  2) X = Make(Ax,k=2); // Make a set of
                       // linguistic
                       // elements with
                       // length k (k=2)
  Repeat
   3) Collect P^k from e_k, p_ij^k ∈ X_le;
   4) R_X = [x_l, x_r] = E_{R_X}(X_le);   // Definition 9
   v(R_X) = [R_X-, R_X+] = [v(R_X(x_l)), v(R_X(x_r))];
   // Definition 10
   5) P^C = [P_ij^C] = Φ(P_ij^k), i,j = 1..n, k = 1..m;
     Where: P_ij^C = [P_ij^C-, P_ij^C+],
   6) Aggregate:
   • P_ij^C- = Φ(p_ij^1-, p_ij^2-, p_ij^3-),  P_ij^C+ = Φ(p_ij^1+, p_ij^2+, p_ij^3+)
   • V^R = [V_i^R]^T = [φ( P_j^C )_i]^T, i,j = 1..n;
     Where: V_i^R = [V_i^R-, V_i^R+],
            V_i^R- = φ( P_j^C- )_i, V_i^R+ = φ( P_j^C+ )_i
   • V^S = [V_1^S, V_2^S, ... V_4^S];
     Where: V_i^S = 1/2 (V_i^R- + V_i^R+)]
   7) Sort(V_i^S), select alternative has the
   highest "satisfied" degree.
  Until(Have no new P^k);
End GDM_HA.
```

IV. Applying Problem

A. Applying problems based the evaluation for the selection of scientific paper

The content in this section presents an applying problem that is solicited from the review panel to select the "best" article published in an international journal. Based on the results of the assessment to give an award to the author of the paper [4]. In this particular GDM model, there are 3 evaluators $E = \{e_1, e_2, e_3\}$, in which f our articles were submitted for evaluation, namely:

$X = \{John's\ paper, Mike's\ paper, David's\ paper, Frank's\ paper\}$.

It can be briefly noted as $X = \{J, M, D, F\}$.

B. Implement the algorithm to solve the GDM

1) Construct computing model

To create the convenience for the experts' evaluations, we need to build a set of linguistic elements as suggestions for the use in the evaluation. We select the structure of the hedge algebra and their fuzzy parameters as:

$G = \{neither < low < medium < high < absolute\}$,
$H = \{H^- = \{Little\} \cup H^+ = \{Very\}\}$,
$fm(low) = 0.5, \alpha = \mu(Little) = 0.5$

2) The linguistic elements which have the maximum length of 2 are generated from hedge algebra:

$$T = \{n < Vl < l < Ll < m < Lh < h < Vh < a\}$$

<u>*Loop:*</u>

3) Collect the evaluated options P^k

The structure of matrix is evaluated as follows:

P^k	J	M	D	F
J	–	P^k_{JM}	P^k_{JD}	P^K_{JF}
M	P^k_{MJ}	–	P^k_{MD}	P^k_{MF}
D	P^k_{DJ}	P^k_{DM}	–	P^k_{DF}
F	P^k_{FJ}	P^k_{FM}	P^k_{FD}	–

Basing on the experts' comments on "more satisfied" among the articles, we obtained the evaluated matrix as follows:

$$P^1 = \begin{bmatrix} - & at\ most\ Vl & Vh & at\ most\ Vl \\ at\ least\ Vh & - & between\ h\ and\ Vh & at\ most\ m \\ l & at\ most\ l & - & greater\ than\ h \\ at\ least\ h & greater\ than\ m & at\ most\ m & - \end{bmatrix}$$

$$P^2 = \begin{bmatrix} - & at\ most\ l & greater\ than\ m & lower\ than\ m \\ greater\ than\ m & - & h & vl \\ at\ most\ Vl & l & - & greater\ than\ h \\ between\ h\ and\ Vh & vh & between\ n\ and\ l & - \end{bmatrix}$$

$$P^3 = \begin{bmatrix} - & greater\ than\ m & between\ h\ and\ Vh & l \\ at\ most\ l & - & at\ least\ h & greate\ than\ m \\ lower\ than\ m & at\ most\ l & - & Vh \\ h & at\ most\ l & vl & - \end{bmatrix}$$

4) Convert the comparative evaluating expressions into the ranges of linguistic elements R_X

We use the transformation function E_{R_X} (Definition 7) to convert. We have the evaluated matrices in which the evaluation options are expressed in terms of linguistic elements.

$$P^1 = \begin{bmatrix} - & [n, Vl] & [Vh, Vh] & [n, Vl] \\ [Vh, a] & - & [h, Vh] & [l, m] \\ [l, l] & [n, l] & - & [Vh, a] \\ [h, a] & [Lh, Vh] & [l, m] & - \end{bmatrix}$$

$$P^2 = \begin{bmatrix} - & [n, l] & [Lh, Vh] & [Vl, Ll] \\ [Lh, Vh] & - & [h, h] & [Vl, Vl] \\ [n, Vl] & [l, l] & - & [Vh, a] \\ [h, Vh] & [Vh, Vh] & [n, l] & - \end{bmatrix}$$

$$P^3 = \begin{bmatrix} - & [Lh, Vh] & [h, Vh] & [l, l] \\ [n, l] & - & [h, a] & [Lh, Vh] \\ [Vl, Ll] & [n, l] & - & [Vh, Vh] \\ [h, h] & [n, l] & [Vl, Vl] & - \end{bmatrix}$$

5) Convert a range of linguistic elements into the semantic ranges

Using the function of linguistic quantification (10) – (13), we can compute the quantitative value of the linguistic elements R_X according to the Definition 8. As the result, we have $[R_{X^-}, R_{X^+}]$ respectively to each evaluation.

$$P^1 = \begin{bmatrix} - & [0, 0.125] & [0.875, 0.875] & [0, 0.125] \\ [0.875, 1] & - & [0.75, 0.875] & [0.25, 0.5] \\ [0.25, 0.25] & [0, 0.25] & - & [0.875, 1] \\ [0.75, 1] & [0.625, 0.875] & [0.25, 0.5] & - \end{bmatrix}$$

$$P^2 = \begin{bmatrix} - & [0, 0.25] & [0.625, 0.875] & [0.125, 0.375] \\ [0.625, 0.875] & - & [0.75, 0.75] & [0.375, 0.375] \\ [0, 0.125] & [0.25, 0.25] & - & [0.875, 1] \\ [0.75, 0.875] & [0.875, 0.875] & [0, 0.25] & - \end{bmatrix}$$

$$P^3 = \begin{bmatrix} - & [0.625, 0.875] & [0.75, 0.875] & [0.25, 0.25] \\ [0, 0.25] & - & [0.75, 1] & [0.625, 0.875] \\ [0.125, 0.375] & [0, 0.25] & - & [0.875, 0.875] \\ [0.75, 0.75] & [0, 0.25] & [0.125, 0.125] & - \end{bmatrix}$$

6) Combination

- Step 1: Use the aggregator Φ to combine the value ranges of the quantifying semantics of the comparative expression by 3 experts. The selected aggregator is the weighted average operator. The result is a matrix P^C, each element $P^C_{ij} = [P^{C-}_{ij}, P^{C+}_{ij}]$ is the range of quantifying semantics that is combined by the 3 ranges quantifying semantics corresponding to the evaluative expressions.

$$P^{C-}_{ij} = \frac{1}{3}\left(p^{1-}_{ij} + p^{2-}_{ij} + p^{3-}_{ij}\right), P^{C+}_{ij} = \frac{1}{3}\left(p^{1+}_{ij} + p^{2+}_{ij} + p^{3+}_{ij}\right)$$

Where: $p^{k-}_{ij} = R_{X^-}{}^{k}_{ij}, p^{k+}_{ij} = R_{X^+}{}^{k}_{ij}$

$$P^C = \begin{bmatrix} - & [0.208, 0.417] & [0.75, 0.875] & [0.125, 0.25] \\ [0.5, 0.708] & - & [0.75, 0.875] & [0.417, 0.58] \\ [0.125, 0.25] & [0.08, 0.25] & - & [0.875, 0.958] \\ [0.75, 0.875] & [0.5, 0.67] & [0.125, 0.292] & - \end{bmatrix}$$

$P^C_{21} = [0.208, 0.417]$

$P^{C-}_{21} = \frac{1}{3}(0 + 0 + 0.625), P^{C+}_{21} = \frac{1}{3}(0.125 + 0.25 + 0.875)$

- Step 2: Use the same aggregator $\varphi \equiv \Phi$ to combine the value range of the quantifying semantics corresponding to the evaluation of a_i to a_l ($i, l = 1..4, l \neq i$). As the resulting, we have the vector $V^R = [V^R_1, V^R_2, \dots V^R_4]$. $V^R_i = [V^{R-}_i, V^{R+}_i]$, the range of quantifying semantics summarized from m experts' judgment of option a_i to all remaining options.

$V^R = [[0.361, 0.514], [0.556, 0.721], [0.360, 0.486], [0.458, 0.612]]$

$V^{R-}_1 = \frac{1}{3}(0.208 + 0.75 + 0.125)$

$V^{R+}_1 = \frac{1}{3}(0.417 + 0.875 + 0.25)$

The results of V^R indicate that quantitative semantics aggregated from the evaluation is "satisfied" by 3 experts to each article (compared to the other articles).

	J	M	D	F
V^R	V^R_1	V^R_2	V^R_3	V^R_4
	[0.361,0.514]	[0.556,0.721]	[0.360,0.486]	[0.458,0.612]

Table 1. Semantic range of the alternatives.

Corresponding to the semantic range $V^R_i = [V^{R-}_i, V^{R+}_i], i = 1..4$, the semantic value V^{R-}_i shows the lowest degree of "satisfied" and V^{R+}_i is the highest degree of "satisfied" of option a_i to all remaining options. It can be called optimistic and pessimistic for each article. Based on Table 1, $V^{R-}_2 = 0.556$ is the lowest degree of "satisfied" and $V^{R+}_2 = 0.721$ is the highest degree of "satisfied". This is average of experts to Mike's article compared to all remaining articles.

	J	M	D	F

Pessimistic	0.361	0.556	0.360	0.458
Optimistic	0.514	0.721	0.486	0.612

Table 2. The degree of satisfaction of each article.

- Step 3: Compute the average value of the semantically quantitative value range for each article.

The satisfaction of each article (Table 2) is aggregated into a semantic range. Looking at these values, we do not seem to have a specific evaluation "satisfied" order between the alternatives. To be able to quantify more specifically, we can compute the average value per interval of each article. We receive V^S.

$V^S = [V_1^S, V_2^S, \dots V_4^S]$. Where: $V_i^S = \frac{1}{2}(V_i^{R-} + V_i^{R+})]$

	$J\ (V_1^S)$	$M\ (V_2^S)$	$D\ (V_3^S)$	$F\ (V_4^S)$
Preference	0.4375	0.6358	0.423	0.535

Table 3. The degree of average satisfaction of each article.

7) Arrange and select the best option

Basing on *Table 3*, we can arrange the "satisfied" degree in descending order of "satisfied" expressed by the semantic value as follows:

$$V_2^S > V_4^S > V_1^S > V_3^S$$

Corresponding to these alternatives, the descending "more satisfied" order of experts to articles is sorted as follows:

$$Mike's\ paper > Frank's\ paper > John's\ paper > David's\ paper$$

Result is Selection 2, Mike's article, has the highest "satisfied" degree summarized from experts' opinion.

- Quantitative semantics range represents the degree of "satisfied":

$$V_2^R = [0.556, 0.721]$$

- The mean value of the quantitative semantics represents the degree of "satisfied":

$$V_2^S = 0.6358$$

<u>*End loop*</u>

Repeat steps 3th to 7th until there are no new or different evaluate from the experts.

Discussion

The article has achieved some main results and new contributions.

- Define the comparatively linguistic conversion function to the ranges of linguistic elements (Definition 7) and the computation of the quantitative semantics values for these ranges (Definition 8).
- Propose algorithms for solving group decision problems to comparative linguistic expressions based on hedge algebra.
- Apply the proposed algorithm in order to solve the group decision problem and the peer reviewers' comparative evaluation using natural language to select the posted article which is highest rated to award the author.

The above results show that the applicability of the hedge algebra in fuzzy problems is expressed based on natural language. Through the problem solved, some recommendations for further study should be,

- When collecting expert opinions, it is necessary to determine the inconsistency between the evaluations of q_i to q_j and of q_j to q_i (i.e. p_{ij}^k is an inconsistency to p_{ji}^k).
- Select a weighted join to be able to meet the priority degree of the experts' evaluation. That is, each evaluated expert carries a different weight. Important experts will carry heavy weight and vice versa.
- Process the evaluation matrices which are lacked of evaluation expressions of experts.

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