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Article

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

J.A.Nannaware[1], Vikas K.Wakode[2], S.S.Phulsagar[3], P.D.Kundgar[4], N Krishnamoorthy[5]

[1,2]Department of PG Studies and Research in Mathematics, Shrikrishna Mahavidyalaya, Gunjoti, Dist.Dharashiv-413606(MS).India
jag_skmg91@rediffmail.com
vikaswakode08@gmail.com
[3]Department of Mathematics, Ramkrishna Paramhansa Mahavidyalaya, Dharashiv-413501(MS).India
ssp.maths@yahoo.co.in
[4]Department of Mathematics,Late N. P. W. Arts and Science College Chopa Dist. Gondia (M.S.) India -441807
pandurang.kundgar30@gmail.com
[5]Faculty of Science and Humanities, PG Department of Computer Applications (MCA), SRM INSTITUTE OF SCIENCE AND TECHNOLOGY, Ramapuram, Chennai 600089, Tamilnadu, India. https://orcid.org/0000-0003-1792-3496.
Email:krishnan@srmist.edu.in

Abstract: In this paper, we investigate the existence and uniqueness of solution of fractional differential equations involving Caputo fractional derivative with nonlocal boundary conditions. The monotone method is constructed using lower and upper solutions. Two monotone sequences are constructed and shown to converge to the minimal and maximal solutions.

Keywords: Fractional differential equations, Existence and uniqueness, Caputo fractional derivative, Lower and upper solutions.
Mathematics Subject Classification (2020): 34A08, 34B15, 34B18.

1. INTRODUCTION

Fractional-order differential equations (FDEs) arise naturally in a wide range of scientific and engineering disciplines, including physics, chemistry, biology, control theory, and many other applied sciences. For instance, Q.T.Ain *et al.* [2] employed the Caputo fractional derivative to formulate a nonlinear model describing the transmission dynamics of the Middle East Respiratory Syndrome Corona virus (MERS-CoV) between humans and camels, where camels were considered the primary source of infection. Their study demonstrated the effectiveness of fractional-order models in capturing the temporal dynamics of infectious diseases.

The theory of fractional differential equations has seen significant development under various types of conditions. The study of existence, uniqueness, stability, and qualitative behavior of solutions for fractional differential equations has become an active area of research. P.W. Eloe *et al.* [6] explored the theory of Caputo fractional differential equations through the study of initial value problems, focusing on the existence, uniqueness, and qualitative analysis of solutions. Z.W. Lv *et al.* [9] investigated fractional differential equations with nonlocal initial conditions in Banach spaces and analyzed the existence and behavior of solutions. R. P. Agarwal *et al.* [1] established sufficient conditions for the existence of solutions to fractional differential inclusions with both convex-valued and nonconvex-valued. M. Benchohra *et al.* [3,4] also established sufficient conditions for the existence of solutions to boundary value problems for fractional differential equations and inclusions involving the Caputo fractional derivative of order $q \in (1,2]$. In particular, they obtained existence results for fractional differential equations subject to nonlinear integral boundary conditions. In (2018), Nora Tabouche *et al.* [17] investigated the existence and uniqueness

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of solutions for Caputo fractional differential equations with nonlocal boundary conditions using the Schauder fixed point theorem and the Banach contraction mapping principle.

The monotone method, based on the upper and lower solutions, has become an effective and constructive method for studying nonlinear fractional differential equations. Lakshmikantham and Vatsala [8] developed a general uniqueness and monotone method for fractional differential equations. Later, F. A. McRae [10] employed monotone method to establish existence results for fractional differential equations. J. D. Ramírez and A. S. Vatsala [15] extended the monotone method to fractional differential equations with periodic boundary conditions, while S. Zhang [20] developed a monotone method for initial value problems involving Riemann–Liouville fractional derivatives. G. Wang *et al.* [18] applied the monotone method to systems of nonlinear fractional differential equations and established corresponding existence results. Moreover, D. Wardowski [19] applied a monotone method to systems of a finite number of nonlinear fractional differential equations and derived sufficient conditions for the existence of solutions. Further contributions were made by J. A. Nanware *et al.* [11,12,13,14] who established the existence and uniqueness of solutions for Riemann–Liouville fractional differential equations with integral boundary conditions. They also proposed a monotone method for systems of Riemann–Liouville fractional differential equations with integral boundary conditions and initial value problems andsubsequently investigated boundary value problems involving the Caputo fractional derivative. These developments demonstrate that the monotone method is a powerful alternative to fixed-point techniques for establishing the existence and uniqueness results for fractional differential equations.

Motivated by the above works, in this paper we investigate the existence and uniqueness of solutions for following Caputo fractional differential equation with nonlocal boundary conditions by using monotone method

$$^{c}D^{q}u(t)+f\big(t,u(t)\big)+{}^{c}D^{q-1}g\big(t,u(t)\big)\ =0, u(\xi)=0,\qquad {}^{c}D^{r}u(1)=\mu u(\eta). \tag{1.1}$$

where $1<q<2$, $0<r<1$, $0<\mu<1$, $0<\eta<\xi<1$, $t\in[0,T]$, $f,g\in C(J\times R,R)$, and $J=[0,T]$. $^{c}D^{q}$ denote the Caputo fractional derivative of order q.

The rest of paper is organized in the following manner:

In section 2, we consider some definitions and preliminary results required for the study. In section 3, using monotone method for the problem is developed. As an application of the monotone method, existence and uniqueness results for Caputo fractional differential equations with nonlocal boundary conditions are obtained. In section 4, at the end conclusion section is added.

2.PRELIMINARIES

In this section, we introduce the definitions and preliminary results that are required for the main results. Define:

i) $C(J,\mathbb{R})=\{u(t)|u(t)$ is continuous on J},

ii) $C_p(J,\mathbb{R})=\{(v(t),w(t))|v(t),w(t)$ are continuous on J, $v(t)\le w(t)$ on $J\}$.

Definition 1. *[7] The Caputo integral of $u(t)$ of fractional order q is defined as* $I^{q}u(t)=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}u(s)\,ds.$

Definition 2. *[7] The Caputo derivative of $u(t)$ of fractional order q is defined as* $^{c}D^{q}u(t)=I^{n-q}u^{(n)}(t)=\frac{1}{\Gamma(n-q)}\int_0^t(t-s)^{n-q-1}u^{(n)}(s)\,ds,$ where $n=[q]+1$.

Definition 3. *A pair of functions v(t) and w(t) in $C_p(J,\mathbb{R})$ are said to be coupled lower and upper solutions of the problem (1.1) if* $^{c}D^{q}v(t)\le -f(t,v(t))-{}^{c}D^{q-1}g(t,w(t)),\quad v(\xi)\le 0,\quad {}^{c}D^{r}v(1)\le\mu v(\eta),$
$^{c}D^{q}w(t)\ge -f(t,w(t))-{}^{c}D^{q-1}g(t,v(t)),\quad w(\xi)\ge 0,\quad {}^{c}D^{r}w(1)\ge\mu w(\eta).$

Lemma 1. *[5] Let* $m \in C_p([t_0,T];\mathbb{R})$. *Suppose that for any* $t_1 \in [t_0,T]$, *we have* $m(t_1) = 0$ *and* $m(t) < 0$, *for* $t_0 \le t < t_1$, *then it follows that* $D^q m(t_1) \ge 0$.

Lemma 2. *[7] For a function* $u \in C^n[0,T]$ *and* $q > 0$, *the following relation hold* $I^q({}^cD^q u(t)) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}$.

Corollary 1. *Let* $m \in C^1(J,\mathbb{R})$ *be such that* ${}^cD^q m(t) \le -Mm(t), \quad m(\xi) \le 0, \quad {}^cD^r m(1) \le \mu m(\eta)$, *for* $0 \le t \le T$ *and* $M > 0$. *Then* $m(t) \le 0$, $0 \le t \le T$.

Similarly, if ${}^cD^q m(t) \ge -Mm(t), \quad m(\xi) \ge 0, \quad {}^cD^r m(1) \ge \mu m(\eta)$, *for* $0 \le t \le T$ *and* $M > 0$. *Then* $m(t) \ge 0, \quad 0 \le t \le T$.

Proof. Proof is analogous to the proof of corollary 2.7 [16].

3.MAIN RESULTS

In this section, we develop the monotone method for the Caputo fractional differential equation (1.1) and establish the existence and uniqueness of the solution to problem (1.1).

THEOREM 1. *Assume that :*
(i) The function f(t,u(t)) and g(t,u(t)) is continuous, f(t,u(t)) is nondecreasing in u for each t and for $u \le v$, ${}^cD^{q-1} g(t,u) = {}^cD^{q-1} g(t,v)$.
(ii) The functions $v_0(t)$ *and* $w_0(t)$ *in* $C_p(J,\mathbb{R})$ *are lower and upper solutions of (1.1) such that* $v_0(t) \le w_0(t)$ *on* J.
(iii) The functions f(t,u) and ${}^cD^{q-1} g(t,u)$ *satisfy one-sided Lipschitz condition*

$$\begin{aligned} f(t,u(t)) - f(t,v(t)) &\le M(u(t) - v(t)), \quad M \ge 0, \\ {}^cD^{q-1} g(t,u(t)) - {}^cD^{q-1} g(t,v(t)) &\le M(u(t) - v(t)), \quad M \ge 0. \end{aligned}$$

Then there exists monotone sequences $v_n(t)$ *and* $w_n(t)$ *in* $C(J,\mathbb{R})$ *such that* $v_n(t) \to v(t)$ and $w_n(t) \to w(t)$ as $n \to \infty$ *and v(t) and w(t) are minimal and maximal solutions of (1.1) respectively.*

Proof. For any ϕ and ψ in $C_p(J,\mathbb{R})$ such that for $v_0(0) \le \phi$ and $w_0(0) \le \psi$ on J, consider the following linear fractional differential equation

$$\begin{aligned} {}^cD^q u(t) + Mu(t) &= -f(t,\phi(t)) - {}^cD^{q-1} g(t,\psi(t)) + M\phi(t), \\ u(\xi) &= 0, \quad {}^cD^r u(1) = \mu u(\eta). \end{aligned} \tag{3.1}$$

We first establish the uniqueness of solution of linear fractional differential equation (3.1). For this, let u_1 and u_2 be two solutions of (3.1). Then we obtain

$$\begin{aligned} {}^cD^q u_1(t) + Mu_1(t) &= -f(t,\phi(t)) - {}^cD^{q-1} g(t,\psi(t)) + M\phi(t), \\ u_1(\xi) &= 0, \quad {}^cD^r u_1(1) = \mu u_1(\eta), \\ {}^cD^q u_2(t) + Mu_2(t) &= -f(t,\phi(t)) - {}^cD^{q-1} g(t,\psi(t)) + M\phi(t), \\ u_2(\xi) &= 0, \quad {}^cD^r u_2(1) = \mu u_2(\eta), \end{aligned}$$

Hence

$${}^cD^q(u_1(t) - u_2(t)) = -M(u_1(t) - u_2(t))$$

and

$$u_1(\xi)-u_2(\xi) = 0,$$
$$ {}^cD^r u_1(1)-{}^cD^r u_2(1) = 0.$$

This implies $u_1(t)=u_2(t)$.
Define the sequences as follows:

$$\begin{aligned} {}^cD^q v_{n+1}(t) &= -f(t,v_n(t))-{}^cD^{q-1}g(t,w_n(t))-M(v_{n+1}-v_n), \\ v_{n+1}(\xi) &= 0, \quad {}^cD^r v_{n+1}(1) = \mu v_{n+1}(\eta), \\ {}^cD^q w_{n+1}(t) &= -f(t,w_n(t))-{}^cD^{q-1}g(t,v_n(t))-M(w_{n+1}-w_n), \\ w_{n+1}(\xi) &= 0, \quad {}^cD^r w_{n+1}(1) = \mu w_{n+1}(\eta). \end{aligned} \tag{3.2}$$

Hence, it follows that the above equation admits a unique solution $v_{n+1}(t)$ and $w_{n+1}(t)$. By setting $n=0$ in equation (3.2), the existence of the solutions $v_1(t)$ and $w_1(t)$ is established. Next we show that $v_0(t)\le v_1(t)\le w_1(t)\le w_0(t)$. Setting $p(t)=v_0(t)-v_1(t)$ we have

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v_0(t)-{}^cD^q v_1(t) \\ &\le -f(t,v_0)-{}^cD^{q-1}g(t,w_0)) \\ &\quad -\{-f(t,v_0)-{}^cD^{q-1}g(t,w_0))-M(v_1-v_0)\} \\ &\le -M(v_0(t)-v_1(t)) \\ &\le -Mp(t). \end{aligned}$$

Hence,

$${}^cD^q p(t) \le -Mp(t).$$

Also,

$$\begin{aligned} p(\xi) &= v_0(\xi)-v_1(\xi) \\ &\le 0 \end{aligned}$$

and

$$\begin{aligned} {}^cD^r p(1) &= {}^cD^r v_0(1)-{}^cD^r v_1(1) \\ &\le \mu(v_0(\eta)-v_1(\eta)) \\ &\le \mu p(\eta). \end{aligned}$$

Thus,

$$\begin{aligned} {}^cD^q p(t) &\le -Mp(t), \\ p(\xi) &\le 0, \\ {}^cD^r p(1) &\le \mu p(\eta). \end{aligned}$$

Hence $v_0(t)\le v_1(t)$. Similarly, it can be shown that $w_0(t)\ge w_1(t)$. Also, we prove $v_1(t)\le w_1(t)$. Setting $p(t)=v_1(t)-w_1(t)$ we have

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v_1(t)-{}^cD^q w_1(t) \\ &\le -f(t,v_0)-{}^cD^{q-1}g(t,w_0)-M(v_1-v_0) \\ &\quad -\{-f(t,w_0)-{}^cD^{q-1}g(t,v_0)-M(w_1-w_0)\} \\ &\le M(w_0-v_0)-M(v_1-v_0)+M(w_1-w_0) \\ &\le -M(v_1(t)-w_1(t)) \\ &\le -Mp(t). \end{aligned}$$

Hence,

$${}^cD^q p(t) \le -Mp(t).$$

Also,

$$\begin{aligned} p(\xi) &= v_1(\xi) - w_1(\xi) \\ &\leq 0, \end{aligned}$$

and

$$\begin{aligned} {}^cD^r p(1) &= {}^cD^r v_1(1) - {}^cD^r w_1(1) \\ &\leq \mu(v_1(\eta) - w_1(\eta)) \\ &\leq \mu p(\eta). \end{aligned}$$

Therefore,

$$\begin{aligned} {}^cD^q p(t) &\leq -Mp(t), \\ p(\xi) &\leq 0, \\ {}^cD^r p(1) &\leq \mu p(\eta). \end{aligned}$$

Hence $v_1(t) \leq w_1(t)$. Thus $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Assume that for some $k > 1$, $v_{k-1}(t) \leq v_k(t) \leq w_k(t) \leq w_{k-1}(t)$. We claim that $v_k(t) \leq v_{k+1}(t) \leq w_{k+1}(t) \leq w_k(t)$ on J. To prove this, set $p(t) = v_k(t) - v_{k+1}(t)$. Since $f(t,u) + Mu$ is nondecreasing in u and $g(t,u)$ is continuous, it follows that

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v_k(t) - {}^cD^q v_{k+1}(t) \\ &= -f(t,v_{k-1}) - {}^cD^{q-1} g(t,w_{k-1}) - M(v_k - v_{k-1}) \\ &\quad -\left\{-f(t,v_k) - {}^cD^{q-1} g(t,w_k) - M(v_{k+1} - v_k)\right\} \\ &\leq M(v_k - v_{k-1}) - M(v_k - v_{k-1}) + M(v_{k+1} - v_k) \\ &\leq -M(v_k(t) - v_{k+1}(t)) \\ &\leq -Mp(t). \end{aligned}$$

Hence,

$${}^cD^q p(t) \leq -Mp(t).$$

Also,

$$\begin{aligned} p(\xi) &= v_k(\xi) - v_{k+1}(\xi) \\ &\leq 0 \end{aligned}$$

and

$$\begin{aligned} {}^cD^r p(1) &= {}^cD^r v_k(1) - {}^cD^r v_{k+1}(1) \\ &\leq \mu(v_k(\eta) - v_{k+1}(\eta)) \\ &\leq \mu p(\eta). \end{aligned}$$

Thus,

$$\begin{aligned} {}^cD^q p(t) &\leq -Mp(t), \\ p(\xi) &\leq 0, \\ {}^cD^r p(1) &\leq \mu p(\eta). \end{aligned}$$

Hence $v_k \leq v_{k+1}$. Similarly we prove that $v_{k+1}(t) \leq w_{k+1}(t)$ Define $p(t) = v_{k+1}(t) - w_{k+1}(t)$ we have

$$
\begin{aligned}
{}^cD^q p(t) &= {}^cD^q v_{k+1}(t) - {}^cD^q w_{k+1}(t) \\
&= -f(t,v_k) - {}^cD^{q-1} g(t,w_k) - M(v_{k+1}-v_k)) \\
&\quad -\{-f(t,w_k) - {}^cD^{q-1} g(t,v_k) - M(w_{k+1}-w_k)\}. \\
&\le M(w_k - v_k) - M(v_{k+1}-v_k) + M(w_{k+1}-w_k) \\
&\le -M(v_{k+1}(t) - w_{k+1}(t)) \\
&\le -Mp(t).
\end{aligned}
$$

Hence,

$$ {}^cD^q p(t) \le -Mp(t). $$

Also,

$$
\begin{aligned}
p(\xi) &= v_{k+1}(\xi) - w_{k+1}(\xi) \\
&\le 0,
\end{aligned}
$$

and

$$
\begin{aligned}
{}^cD^r p(1) &= {}^cD^r v_{k+1}(1) - {}^cD^r w_{k+1}(1) \\
&\le \mu(v_{k+1}(\eta) - w_{k+1}(\eta)) \\
&\le \mu p(\eta).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
{}^cD^q p(t) &\le -Mp(t), \\
p(\xi) &\le 0, \\
{}^cD^r p(1) &\le \mu p(\eta).
\end{aligned}
$$

Hence $$v_{k+1} \le w_{k+1}.$$

Thus $$v_0 \le v_1 \le v_2 \le \cdots \le v_k \le v_{k+1} \le w_{k+1} \le w_k \le \cdots \le w_1 \le w_0.$$

Using corresponding fractional volterra integral equations

$$
\begin{aligned}
v_{n+1}(t) = a_n + b_n t + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{ &-f(s, v_{(n+1)}(s) - {}^cD^{q-1} g(t, w_{n+1}(s)) \\
&-M(v_{n+1}(s) - v_n(s))\} ds,
\end{aligned}
$$

$$
\begin{aligned}
w_{n+1}(t) = c_n + d_n t + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{ &-f(s, w_{(n+1)}(s) - {}^cD^{q-1} g(t, v_{n+1}(s)) \\
&-M(w_{n+1}(s) - w_n(s))\} ds
\end{aligned}
$$

it follows that $v(t)$ and $w(t)$ are solutions of problem (3.1). Next, we prove that $v(t)$ and $w(t)$ are the minimal and maximal solutions of (1.1). Let $u(t)$ be any solution of (1.1) different form $v(t)$ and $w(t)$, so that there exists k such that $v_k(t) \le u_k(t) \le w_k(t)$ on J and set $p(t) = v_{k+1}(t) - u_k(t)$ we have

$$
\begin{aligned}
{}^cD^q p(t) &= {}^cD^q v_{k+1}(t) - {}^cD^q u_k(t) \\
&= -f(t,v_k) - {}^cD^{q-1} g(t,w_k) - M(v_{k+1}-v_k) \\
&\quad -\{-f(t,u_k) - {}^cD^{q-1} g(t,u_k)\} \\
&\le M(u_k - v_k) - M(v_{k+1}-v_k) \\
&\le -M(v_{k+1}(t) - u_k(t)) \\
&\le -Mp(t).
\end{aligned}
$$

Hence,

$$ {}^cD^q p(t) \le -Mp(t). $$

Also,

$$\begin{aligned} p(\xi) &= v_{k+1}(\xi) - u_k(\xi) \\ &\le 0 \end{aligned}$$

and

$$\begin{aligned} {}^cD^r p(1) &= {}^cD^r v_{k+1}(1) - {}^cD^r u_k(1) \\ &\le \mu(v_{k+1}(\eta) - u_k(\eta)) \\ &\le \mu p(\eta). \end{aligned}$$

Thus,

$$\begin{aligned} {}^cD^q p(t) &\le -Mp(t), \\ p(\xi) &\le 0, \\ {}^cD^r p(1) &\le \mu p(\eta). \end{aligned}$$

Thus $v_{k+1} \le u_k(t)$ on $[0,T]$. Since $v_0(t) \le u_0(t)$ on $[0,T]$, by induction it follows that $v_k(t) \le u_k(t)$ for all k. Similarly, we show that $u_k(t) \le w_k(t)$ for all k on $[0,T]$. Set $p(t) = u_k(t) - w_{k+1}(t)$ we have

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q u_k(t) - {}^cD^q w_{k+1}(t) \\ &= -f(t,u_k) - {}^cD^{q-1} g(t,u_k) \\ &\quad - \{-f(t,w_k) - {}^cD^{q-1} g(t,v_k) - M(w_{k+1} - w_k)\} \\ &\le M(w_k - u_k) + M(w_{k+1} - w_k) \\ &\le -M(u_k(t) - w_{k+1}(t)) \\ &\le -Mp(t) \end{aligned}$$

Hence,

$${}^cD^q p(t) \le -Mp(t).$$

Also,

$$\begin{aligned} p(\xi) &= u_k(\xi) - w_{k+1}(\xi) \\ &\le 0, \end{aligned}$$

and

$$\begin{aligned} {}^cD^r p(1) &= {}^cD^r u_k(1) - {}^cD^r w_{k+1}(1) \\ &\le \mu(u_k(\eta) - w_{k+1}(\eta)) \\ &\le \mu p(\eta). \end{aligned}$$

Therefore,

$$\begin{aligned} {}^cD^q p(t) &\le -Mp(t), \\ p(\xi) &\le 0, \\ {}^cD^r p(1) &\le \mu p(\eta). \end{aligned}$$

Hence $u_k(t) \le w_{k+1}(t)$. By induction it follows that $u_k(t) \le w_k(t)$ for all k. Thus $v_k(t) \le u_k(t) \le w_k(t)$ on $[0,T]$. Taking limit as $n \to \infty$, it follows that $v(t) \le u(t) \le w(t)$ on $[0,T]$. □

Next we obtain the uniqueness of solution of (1.1) in the following:

THEOREM 2. *Assume that :*

(i) The function f(t,u(t)) and g(t,u(t)) is continuous, f(t,u(t)) is nondecreasing in u for each t and for $u \le v$, ${}^cD^{q-1} g(t,u) = {}^cD^{q-1} g(t,v)$.

(ii) The functions $v_0(t)$ *and* $w_0(t)$ *in* $C_p(J,\mathbb{R})$ *are lower and upper solutions of (1.1) such that* $v_0(t) \le w_0(t)$ *on* J.

(iii) The functions f(t,u) and ${}^cD^{q-1}g(t,u)$ *satisfy Lipschitz condition*

$$\begin{aligned} |f(t,u(t)) - f(t,v(t))| &\le M|u(t) - v(t)|, \qquad M \ge 0, \\ \left|{}^cD^{q-1}g(t,u(t)) - {}^cD^{q-1}g(t,v(t))\right| &\le M|u(t) - v(t)|, \qquad M \ge 0. \end{aligned}$$

Proof. Since $v(t) \le w(t)$, we need to prove $v(t) \ge w(t)$. Consider $p(t) = w(t) - v(t)$ we find that

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q w(t) - {}^cD^q v(t) \\ &= -f(t,w) - {}^cD^{q-1}g(t,v) - \left\{-f(t,v) - {}^cD^{q-1}g(t,w)\right\} \\ &\le -M(w(t) - v(t)) \\ &\le -Mp(t) \end{aligned}$$

Hence,

$$ {}^cD^q p(t) \le -Mp(t). $$

and

$$\begin{aligned} {}^cD^r p(1) &= {}^cD^r w(1) - {}^cD^r v(1) \\ &\le \mu(w(\eta) - v(\eta)) \\ &\le \mu p(\eta). \end{aligned}$$

Thus,

$$\begin{aligned} {}^cD^q p(t) &\le -Mp(t), \\ p(\xi) &\le 0, \\ {}^cD^r p(1) &\le \mu p(\eta). \end{aligned}$$

Thus, it follows that $p(t) \le 0$. This gives $w(t) \le v(t)$. Hence $v(t) = w(t)$. Thus, there exists unique solution of problem (1.1) on [0,T].

4. CONCLUSION

The existence and uniqueness of solutions for Caputo fractional differential equations with nonlocal boundary conditions were investigated by employing the monotone method of lower and upper solutions. Furthermore, two monotone sequences were constructed that converges to the minimal and maximal solutions of problem (1.1).

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