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Article

The Geo Johan Chromatic Number and Its Boundaries for Various of Graphs

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Abstract:A collection S of vertices from this set is referred to as a g(G) when every vertex present in the graph can be found lying along at least one most direct routeconnecting some pair of vertices drawn from S. The smallest possible size of such a collection is known as the geodetic number, written as g(G). Separately, a legitimate vertex colouring of a graph is described as a Johan colouring when each and every vertex in the graph possesses what is termed a rainbow neighbourhood — meaning that among all vertices adjacent to a given vertex, every colour used in the colouring appears at least once. The largest count of distinct colours that can be employed within any such valid colouring is referred to as the Johan chromatic number, denoted J(G). Building upon these two foundational ideas, this work puts forward a unified concept called the geodetic Johan chromatic set. A vertex subset qualifies as a geodetic Johan chromatic set only when it simultaneously satisfies the conditions required of both a g(G) and a J(G). The lowest cardinality achievable by any such combined set defines a new graph parameter called the Geo Johan chromatic number, represented by the notation $J_{gc}(G)$. This study systematically derives the value of $J_{gc}(G)$ across a variety of well-known and standard families of graphs. Beyond these specific computations, the paper also establishes tight and sharp bounds that govern this parameter for the broader family of connected graphs. Furthermore, one particularly noteworthy finding demonstrated in this work is that the geodetic Johan chromatic number does not behave monotonically with respect to the subgraph relationship — that is, moving to a subgraph does not necessarily decrease or preserve this number in a predictable direction,thesmallest cardinality of any such set is designated the Geo Johan chromatic number, expressed as$J_{gc}(G)$. The value of$J_{gc}(G)$is determined for various standard graph families, and sharp bounds are established for connected graphs. Furthermore, it is demonstrated that the geodetic Johan chromatic number fails to be monotone under the subgraph relation.
Keywords: Geodetic number, Johan Colouring, Geo chromatic number, Geo Johan Chromatic number

1. Introduction

Throughout this paper, attention is restricted exclusively to graphs that are finite, simple, and connected in nature, with no loops or multiple edges permitted between any pair of vertices, and all standard terminology and notation that are not explicitly defined within this work follow the conventions established in the foundational literature [10]. As Buckley and Harary established that this distance measure is central to a wide class of optimization problems on graphs [1]. Chartrand, Harary, and Zhang, who used this closure operator to characterize the structure of minimal g(G) across various graph families [2]. When this closure expands to cover the entire vertex set, meaning I[S] is equal toV(G), the subset S is classified as a g(G), and among all possible geodetic sets, the one with the fewest vertices determines the geodetic number g(G), which was systematically introduced and analysed for paths, cycles, trees, and complete graphs by Chartrand and Zhang, who also established early relationships between the geodetic number and other distance-based parameters [3]. The behaviour of the geodetic number under graph products such as the Cartesian product and the strong product was further investigated by Klavžar and Mulder, who showed that product structures can dramatically affect the size of lowest geodetic sets and require specialized arguments [4]. Computing g(G) is NP-hard for general graphs while remaining tractable for specific structured families such as chordal graphs [6]. Geodetic sets in trees were characterized by Harary and Loukakis, who showed that the leaves of a tree always belong to every lowest geodetic set, providing a clean structural characterization that does not extend directly to more general graph classes [7]. The upper geodetic number, was introduced and studied by Chartrand et al., revealing an asymmetry between lower and upper bounds that mirrors similar distinctions in domination theory [8], while the forcing geodetic number, which captures how uniquely a lowest geodetic set is determined, was examined by Zhang, who showed connections between forcing numbers and the symmetry properties of the underlying graph [9]. Properties of geodetic sets in bipartite graphs were further examined by Atici, who showed that bipartiteness imposes structural constraints that can reduce the geodetic number relative to comparable non-bipartite graphs of the same order [15], and the relationship between the geodetic number and graph diameter was explored by Hernando et al., who proved that in certain graph families the geodetic number can be bounded effectively using diameter and other distance parameters [13].

Concerning vertex colouring, a proper p-colouring when its vertices can be assigned colours, representing the smallest such value of p, along with its relationship to graph structure, was studied extensively by Jensen and Toft, who compiled a wide collection of colouring results establishing chromatic number as one of the most central parameters in all of graph theory [12]. An alternative colouring perspective is introduced through the concept of reverse colouring, achieved by applying the mapping that interchanges colour class p_i with colour class $p_{\chi-i+1}$, effectively reversing the colourlabelling scheme to produce what is referred to as the reverse c-colouring of the graph. V(G) is said to have a rainbow neighbourhood when its closed neighbourhood N[v] is non-empty for every index i ranging from 1 through c(G), as studied by Kok and Sudev, who derived conditions under which standard colouring can be transformed into rainbow-neighbourhood-respecting colourings [14]. The structural role of closed neighbourhoods in defining colouring-based parameters was further analysed by Kok, who demonstrated that the condition of every closed neighbourhood intersecting every colour class imposes strong restrictions on the colouring and yields a parameter distinctly different from classical chromatic numbers [18]. The study of maximal proper colourings, in which no additional color class can be added without violating the proper colouring condition, was discussed by Mitchem and Slater, who investigated the relationship between maximally and chromatic efficiency in graph colouring [16]. The concept of Johan colouring was introduced by Johan Kok and is characterized as a maximal proper vertex colouring of G in which every single vertex of the graph belongs to a rainbow neighbourhood, for which foundational results across standard graph families including complete graphs, paths, and cycles were derived as part of his original investigation [11].

The interaction between geodetic sets and dominating sets was studied by Sampath Kumar and PushpaLatha, leading to the introduction of the geodetic domination number as a combined parameter that simultaneously satisfies coverage and distance conditions [17], reflecting a broader productive trend of combining two independent graph parameters into a single unified concept, as seen also in the domination chromatic number and related hybrid parameters that reveal non-trivial interactions beyond what either parameter captures alone. Motivated by these two independent but complementary notions of geodetic sets and Johan colorings, this paper introduces the geodetic Johan chromatic set as a vertex subset that simultaneously qualifies as both a g(G) and a J(G), whose smallest cardinality is defined as $J_{gc}(G)$. The paper proceeds to determine this parameter for several well-known graph families, derive meaningful bounds for connected graphs, and present general structural results associated with this newly defined concept, including the significant observation that $J_{gc}(G)$ does not behave monotonically under the subgraph relation.

2. GEO JOHAN CHROMATIC NUMBER

A connected graph G's subset $S \subseteq V(G)$ is called a Geo Johan chromatic set.if:

• S constitutes a g(G); that is, Each vertex in G is located on a most direct routethat joins a pair of vertices in S.

• S supports a Johan colouring of G, meaning that all colour classes are represented collectively by the vertices of S in a Johan colouring of G. The Geo Johan chromatic number of G, is the smallest cardinality among all $J_{gc}(G)$. Formally, $J_{gc}(G) = \min\{ |S|: S \subseteq V(G) \text{ is a } J_{gc}(G) \}$.

Remarks 1:

- Every Geo Johan chromatic set is necessarily both a g(G) and a $J_{gc}(G)$, but the converse need not hold.
- Not all connected graphs admit a geo-Johan chromatic set.

Examples: For a connected graph G illustrated in Fig. 1, G is 3-colorable, admits a Johan colouring employing three colours. The set $S = \{v_1, v_5, v_6, v_7\}$ constitutes a leastg(G). each vertex in G is located on a most direct routeconnecting a pair of vertices in S. Hence g(S) = 4. Nevertheless, S does not qualify as a Geo-Johan chromatic set, given that it fails to include vertices from all colour classes of the Johan colouring. To construct a Geo-Johan chromatic set, it is necessary to incorporate an additional vertex corresponding to the absent colour class. Thus $J_{gc}(G) = S \cup \{v_4\}$ forms a Geo-Johan chromatic set of G. Consequently, $J_{gc}(G) = 5$.

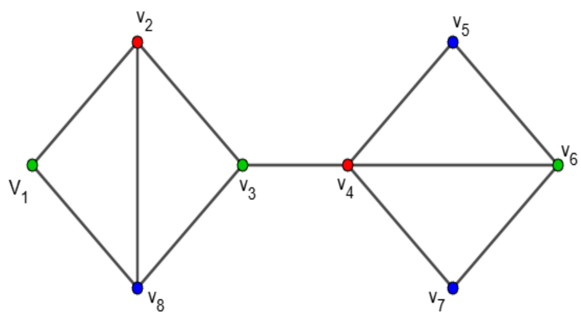

Figure 1: The Graph G withJ_{gc} (G) = 5.

3. RESULTS

Theorem 1. For the path graph P_n, $n > 1$, the geodetic Johan chromatic number is $J_{gc}(P_n)=2$ for every $n > 1$.

Proof: Let $V(P_n) = \{v_1, v_2,\ldots..,v_n\}$with edges v_iv_{i+1}, $1 \leq i< n$. $S = \{v_1, v_n\}$ is a minimum geodetic set of P_n, since P_n is itself the unique shortest v_1–v_n path and every vertex lies on it. v_1 (and v_n) has degree 1, so $N[v_1] = \{v_1, v_2\}$ and $|N[v_1]| = 2$. Since v_1 must yield a rainbow neighbourhood in a Johan colouring, the colour set restricted to $N[v_1]$ must equal the entire colour class set used on G, so $|C| \leq |N[v_1]| = 2$. Hence no Johan colouring of P_n can use more than 2 colours, for any n.

Colour alternately:$c(v_i) = c_1$ if i is odd, c_2 if i is even. This is proper since consecutive vertices differ in parity. For the leaves: $N[v_1] = \{v_1, v_2\} = \{c_1,c_2\}$ — rainbow. $N[v_n]$ similarly rainbow, whichever parity n has. For any internal vertex v_i ($1<i< n$): $N[v_i] = \{v_{i-1}, v_i, v_{i+1}\}$; v_{i-1} and v_{i+1} share the same colour (both differ in parity from v_i), so $N[v_i]$ contains both c_1 and c_2 — rainbow.

Thus every vertex of P_n yields a rainbow neighbourhood under this 2-colouring, so it is a Johan colouring, and the bound above shows 2 is also maximal. Therefore J_{gc}(P_n) = 2 for every n > 1.

Theorem: Let cycle graph Cn, n > 3: $$J_{gc}(C_n)=\begin{cases} 3, & n \equiv 0 (\text{mod } 3) \\ 2 & n \quad even, n \; not \equiv 0 (\text{mod } 3) \\ undefined \; (no \quad Johan \; colouring exists), & n \; not \equiv 0 (\text{mod } 3). \end{cases}$$

Proof: If n is even, $S = \{v_1, v_{1+\frac{n}{2}}\}$ is a valid 2-vertex geodetic set (both arcs between the antipodal pair together cover $V(C_n)$). If n is odd, $S = \{v_1, v_{k+1}, v_{k+2}\}$ ($n=2k+1$) is a valid 3-vertex geodetic set, as verified above.

Since $\delta(C_n) = 2$, no Johan colouring can use more than $\delta + 1 = 3$ colours. A 3-colouring achieving a rainbow closed neighbourhood at every vertex forces, for each i, colour (v_{i+1}) to be the unique colour distinct from colour (v_{i-1}) and colour (v_i); iterating this recurrence produces the strict period-3 sequence $c_1, c_2, c_3, c_1, c_2, c_3,\ldots$, which closes consistently around the cycle exactly when 3, n $\equiv (\text{mod } 3)$. When 3, n $\equiv (\text{mod } 3)$, assign colour (v_i) by i mod 3; this is proper and every $N[v_i]$ contains all three colours, including the geodetic set's vertices, which (being spread around the cycle) automatically receive distinct colours from this construction. When n is even and 3, n not$\equiv (\text{mod } 3)$, the alternating 2-colouring is proper and gives every vertex — including both members of S — a rainbow closed neighbourhood. When n is odd and 3, n not$\equiv (\text{mod } 3)$, neither 2 nor 3 colours can produce a full rainbow colouring (2 colours fail properness since $\chi(C_n) = 3$; 3 colours fail the periodicity requirement above), so no Johan colouring of C_n exists, and $J_{gc}(C_n)$ is undefined for these n.

Remark 2. For n = 3, the cycle C_3 admits a 3-coloring in which every vertex has a rainbow

neighbourhood. Moreover, the vertex set $S = \{v_1, v_2, v_3\}$ forms a lowest geodetic set. Hence $J_{gc}(C_3) = 3$.

Remark 3. For odd cycles C_n with $n > 3$, although lowest geodetic sets exist, no Johan colouring is possible, since odd cycles are neither bipartite nor satisfy the rainbow neighbourhood condition. Consequently, no geodetic Johan colouring exists for such graphs.

Theorem: For complete graph K_n, $n \geq 3$. The geodetic Johan chromatic number of K_n is $J_{gc}(K_n) = n$.

Proof: Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Since K_n is complete, Each unique vertex pair is next to the others.. First, observe that in K_n, no vertex lies on a geodesic between two other distinct vertices, as every most direct routehas length one. Hence, $g(K_n)$ is the entire vertex set $V(K_n)$. Therefore the lowest geodetic set S satisfies $g(K_n) = |S| = n$. Now consider a proper colouring of K_n. Since every pair of vertices is adjacent, a proper colouring requires each vertex to receive a distinct colour. Thus exactly n colours are used, any vertex $v \in V(K_n)$, closed neighbourhood satisfies $N[v] = V(K_n)$. Hence N[v] contains vertices from all colour classes, and every vertex possesses a rainbow neighbourhood. Consequently, every proper colouring of K_n is a Johan colouring. Since no proper colouring of K_n can use more than n colours, it follows that $J_{gc}(K_n) = n$.

Theorem. For n>2, let $K_{1,n-1}$ be the star graph. Then $J_{gc}(K_{1,n-1}) = n$.

Proof: Let $V(K_{1,n-1})=\{v\}\cup\{u_1,u_2,\ldots,u_{n-1}\}$, where v is the central vertex and $u_1,\ldots,u_{n-1}$ are the pendant (leaf) vertices.

The J – chromatic number of $K_{1,n-1}$ is forced to be 2. Each pendant vertex u_i has closed neighbourhood $N[u_i] = \{v, u_i\}$, which has only two elements. For u_i to have a rainbow neighbourhood $N[u_i]$ must contain a vertex of every colour used in G — but a 2 – element set can hold at most 2 distinct colours. Hence no J – colouring of a star (with $n - 1 \geq 2$ leaves) can use more than 2 colours, and since $K_{1,n-1}$ is bipartite, $\chi(K_{1,n-1})=2$ is achievable. So $J(K_{1,n-1})=2$. Because every u_i is adjacent to v, properness forces v and every u_i into different colour classes. With only 2 colours available, all leaves must take the same colour (say c_2), while v takes the other colour c_1 — this is, up to swapping colour labels, the unique J – colouring. So in any J – colouring, the two colour classes are exactly {v} and $\{u_1,\ldots,u_{n-1}\}$.

The geodetic number of $K_{1,n-1}$ is $n - 1$, and the minimum geodetic set is unique. A vertex x is extreme if the subgraph induced on its neighbours is complete; extreme vertices must belong to every geodetic set (standard fact: an extreme vertex cannot lie internally on any shortest path between two other vertices, so it can only be "covered" by being in S itself). Each leaf u_i has a single neighbour {v}, which trivially induces a complete graph, so every leaf is an extreme vertex. Hence every geodetic set of $K_{1,n-1}$ must contain all $n - 1$ leaves.

Conversely, $S = \{u_1,\ldots,u_{n-1}\}$ *is* a geodetic set: for any two leaves u_i,u_j, the unique shortest $u_i - u_j$ path is $u_i - v - u_j$, so v lies on this geodesic and is covered; every leaf trivially covers itself. Thus $I(S) = V(K_{1,n-1})$.

Since every geodetic set must contain all $n - 1$ leaves, and this set of $n - 1$ leaves already suffices, $g(K_{1,n-1}) = n - 1$, and $S = \{u_1,\ldots,u_{n-1}\}$ is the unique minimum geodetic set (the centre v is not extreme, so it need not — and in the minimum set, does not — belong).

Why n−1 vertices are never enough for J_{gc}. Any candidate set S^* for $J_{gc}(K_{1,n-1})$ must be:

1. a geodetic set — by Step 2, this forces $\{u1,\ldots,un-1\}\subseteq S^*$;
2. a representative set for both colour classes of the (unique) J – colouring — by Step 1, the colour classes are {v} and $\{u_1,\ldots,u_{n-1}\}$, so S^* needs at least one vertex from each, i.e. it needs some uiu_iui (already guaranteed) and the vertex v.

Condition (1) alone already forces $|S^*| \geq n-1$, and by itself the leaf-only set of size n−1 satisfies (1) but fails (2), since it contains no vertex of colour c_1 (only v has that colour). Therefore no set of size n−1can satisfy both conditions — every set of size $\leq n - 1$ either misses a leaf (violating geodetic coverage) or, if it is exactly the leaf set, misses the colour class {v} (violating the chromatic requirement). These rules out every set of cardinality less than n, not merely the specific construction used earlier. Take $S^* = \{u_1,\ldots,u_{n-1}\}\cup\{v\}=V(K_{1,n-1})$, with $|S^*| = n$. It is geodetic (contains the unique minimum geodetic set) and it contains a representative of each of the 2 colour classes (v for c_1, any u_i for c_2), so it is a valid Geo Johan chromatic set. Combining these $J_{gc}(K_{1,n-1}) = n$.

Theorem. Let $DS_{m,n}$ be the double star graph, m, n>1. Then $J_{gc}(DS_{m,n}) = m + n$.

Proof: Let $DS_{m,n}$ arise by joining the centres u_0 and v_0 of two stars $K_{1,m}$ and $K_{1,n}$ by an edge. So

$V(DS_{m,n}) = \{u_0,v_0\} \cup \{u_1,\ldots,u_m\} \cup \{v_1,\ldots,v_n\}$, with u_0 adjacent to $u_1,\ldots,u_m$, v_0 adjacent to $v_1,\ldots,v_n$, and $u_0v_0 \in E(DS_{m,n})$. $J(DS_{m,n})=2$, forced by the leaves. Every leaf u_i has $N[u_i]=\{u_0,u_i\}$ (and similarly $N[v_j]=\{v_0,v_j\}$) — only 2 vertices. For u_i to have a rainbow neighbourhood, $N[u_i]$ must contain a vertex of every colour used in G; a 2 – element set cannot represent more than 2 colours. So no J – colouring of $DS_{m,n}$ can use more than 2 colours. Since $DS_{m,n}$ is a tree (hence bipartite), $\chi(DS_{m,n}) = 2$ is achievable, giving $J(DS_{m,n}) = 2$. The correct (unique) 2 – colouring. With only 2 colours and $u_0v_0 \in E$, properness forces $c(u_0) \neq c(v_0)$. Since u_0 is adjacent to every u_i, each u_i must take u_0's opposite colour — which is v_0's colour. Symmetrically, each v_j must take v_0's opposite colour, which is u_0's colour. So the (essentially unique) 2-colouring is: $C_1=\{u_0\} \cup \{v_1,\ldots,v_n\}$, $C_2=\{v_0\}\cup\{u_1,\ldots,u_m\}$.

Verification this is a valid J-colouring:

- $N[u_i]=\{u_0,u_i\}$: $u_0 \in C_1$, $u_i \in C_2$ — both colours present.
- $N[v_j]=\{v_0,v_j\}$: $v_0 \in C_2$, $v_j \in C_1$ — both colours present.
- $N[u_0]=\{u_0,v_0,u_1,\ldots,u_m\}$: contains $u_0 \in C_1$ and v_0, $u_i \in C_2$ — both colours present.
- $N[v_0]$: symmetric — both colours present.

So every vertex has a rainbow neighbourhood; this is a genuine J – colouring with 2 colours, matching $J(DS_{m,n}) = 2$ exactly (this is what the original draft's colouring failed to be).

A vertex is extreme if its neighbourhood induces a complete subgraph; extreme vertices must lie in every geodetic set. Each leaf u_i has the single neighbour $\{u_0\}$ (trivially complete), so every u_i (and every v_j) is extreme. Hence every geodetic set must contain all m + n leaves. Neither u_0 nor v_0 is extreme (e.g. u_0's neighbours $\{v_0,u_1,\ldots,u_m\}$ do not induce a complete graph, since the u_i's are pairwise non-adjacent), so the centres need not belong to a minimum geodetic set.

Conversely, $S=\{u_1,\ldots,u_m,v_1,\ldots,v_n\}$ is geodetic: for any $u_i,v_j \in S$, the unique shortest path is $u_i – u_0 – v_0 – v_j$, which passes through and thus covers both centres. So $I(S) = V(DS_{m,n})$.

Since every geodetic set must contain all m + n leaves, and this set of size m + n already suffices, $g(DS_{m,n}) = m + n$, and S is the unique minimum geodetic set.

Unlike a single star (where all leaves land in one colour class and the geodetic set must be enlarged with the centre), here the two colour classes are $C_1 = \{u_0\} \cup \{v_j\text{'s}\}$ and $C_2 = \{v_0\} \cup \{u_i\text{'s}\}$. The set $S = \{u_i\text{'s}\} \cup \{v_j\text{'s}\}$ already contains vertices of both classes: the u_i's represent C_2 and the v_j's represent C_1. So S is simultaneously geodetic and a full colour-class representative set (chromatic set) for the J-colouring — with no need to add u_0 or v_0

Any candidate set S* for $Jgc(DS_{m,n})$ must be geodetic, forces $S^* \supseteq \{u_1,\ldots,u_m,v_1,\ldots,v_n\}$, i.e. $|S^*| \geq m + n$. Since this lower bound is already attained by S itself, it is best possible. Combining these $J_{gc}(DS_{m,n}) = m + n$.

Theorem: Let $K_{1,n,n}$be the bistar graph, where n > 1. Thus,$J_{gc}(K_{1,n,n}) = n + 2$.

Proof: Let $G = K_{1,n,n}$be obtained from $K_{1,n}$ by inserting one new vertex into each pendant edge. Let$V(G) = \{v\}\cup\{u_1, u_2, \ldots,u_n\}\cup\{w_1, w_2,\ldots, w_n\}$, where v is the central vertex, u_i are the pendant vertices, and $w_i$subdivides the edge vu_i. In G, every most direct routebetween two distinct pendant vertices $u_i$and $u_j$passes through u_i - w_i - v - w_j - u_j. As a result, each vertex is on a geodesic that connects a pair of pendant vertices. The lowest$g(G)$ is $S^* = \{v, w_1, \ldots, w_n\}$, with $|S^*| = n+1$. However, theJohan colouring condition forces three distinct colour classes across u_i, w_i, and v. Including onependant vertex $u_j$yields a set of cardinality n + 2 that satisfies both conditions. Therefore $J_{gc}(K_{1,n,n}) = n + 2$.

Theorem: Let $W_n$be the wheel graph with n > 3 vertices, where n is odd. The geodetic Johan

Chromatic number of $W_n$is $J_{gc}(W_n) = \left\lceil \frac{n}{2} \right\rceil +2$.

Proof: Let $V(W_n) = \{v_0\} \cup \{v_1, v_2, \ldots, v_n\}$, where hub vertex$v_0$ and $\{ v_1, v_2, \ldots, v_n \}$ induces the cycle C_n .Since n is odd, a lowestg(C_n) is $S_1 = \{v_i : i \equiv 0 \pmod 2\}$ with $|S_1| = \left\lceil \frac{n}{2} \right\rceil$.

Each rim vertex is on the most direct routeconnecting two S_1 vertices, and every most direct routebetween non-adjacent rim vertices passes through the hub v_0. Including the hub v_0 and one additional rim vertex $v_k \notin S_1$, and defining $S = S_1 \cup \{v_k\} \cup \{v_0\}$, yields a geodetic Johan chromatic set. Sinceneither the hub nor the additional rim vertex can be omitted without violating the Johan condition, S is minimal. Hence, $J_{gc}(W_n) = |S| = \left\lceil \frac{n}{2} \right\rceil +2$.

Remark 4: For n = 4, the wheel graph W_4admits a lowest geodetic set $S = \{v_0, v_1, v_2, v_3\}$, which also satisfies the Johan colouring condition. Hence $J_{gc}(W_4) = 4$.

4. BOUNDS FOR GEO JOHAN CHROMATIC NUMBER.

Theorem: If $n \geq 2$ for a connected graph,Then$2 \leq J_{gc}(G) \leq n$, both boundaries are sharp.

Proof: Let $n \geq 2$ for a connected graph G.

Lower bound: since G is linked and has at least two vertices. By definition, every geodetic Johan chromatic set is also a g(G). Therefore $J_{gc}(G) \geq g(G) \geq 2$.

Upper bound: Let S be a lowest$J_{gc}(G)$. Clearly, $|S| \leq |V(G)| = n$. Hence $J_{gc}(G) \leq n$.

Sharpness: The path graph reaches the lowest limit, with even n, and the cycle $C_n$with $n \equiv 2 \pmod 4$, where $J_{gc}(G)=2$. The upper bound are attained by K_n and$K_{1,n-1}$, for which $J_{gc}(G) = n$. Thus both bounds are sharp.

Theorem: If $n \geq 2$ for a connected graph. Then $2 \leq g(G) \leq J_{gc}(G) \leq n$.

Proof:Any connected graph of order at least two has a geodetic number, $g(G) \geq 2$. By definition, a geodetic Johan chromatic set is a g(G) that additionally satisfies the Johan colouring condition. Hence every geodetic Johan chromatic set must contain at least g(G) vertices, implying $g(G) \leq J_{gc}(G)$. Finally, since $J_{gc}(G)$ is the cardinality of a subset of V(G), $J_{gc}(G) \leq n$. Combining these inequalities yields $2 \leq g(G) \leq J_{gc}(G) \leq n$.

Theorem: For a connected graph, there exist subgraphs $H \subseteq G$ such that $J_{gc}(H) > J_{gc}(G)$,andthere also exist subgraphs $H \subseteq G$ such that $J_{gc}(H) < J_{gc}(G)$.

Proof: Both statements are established by construction.

Case 1:$J_{gc}(H) > J_{gc}(G)$

Let $G=P_4$, the path on four vertices. From earlier results, $J_{gc}(P_4)=2$.

Let $H = C_3$, subgraph of G created by joining two internal vertices with an edge. Then $J_{gc}(C_3) = 3$. Hence, $J_{gc}(H) > J_{gc}(G)$.

Case 2:$J_{gc}(H) < J_{gc}(G)$.

Let $G=K_n$, $n \geq 3$. Then $J_{gc}(K_n) = n$. Let $H = P_n$ be a spanning subgraph of K_n. For even n, $J_{gc}(P_n)=2$. Thus $J_{gc}(H) < J_{gc}(G)$.

Since both inequalities occur for suitable subgraphs, the geodetic Johan chromatic number is not monotone with respect to the subgraph relation.

Conclusion and Future Scope:In this paper, the Geo Johan chromatic number $J_{gc}(G)$ has been

examined. Bounds and general results for the geo Johan chromatic number have been presented. The concept of geo Johan colouring can be extended to explore total geo Johan chromatic number, open geo Johan chromatic number, and related parameters.

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