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Article

A Soft Set Theoretic Framework for the Structured Classification of Fuzzy Semigroup Subsets

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Abstract: The framework of soft sets for ordered classification of fuzzy semigroups subsets is initiated in this paper. We define two new classes of fuzzy subsets in a semigroup using soft sets operations and investigate the algebraic properties with respect to their union and intersection. We also obtain the relatedness of soft set conditions with certain classes of fuzzy subsets in semigroups, viz.,fuzzy soft sub(ideals/filters). This exercise gives a common framework for such type of complicated constructions which appeared in the context of fuzzy semigroup theory. Key Words: Soft set theory, fuzzy semigroup, fuzzy soft subsets, algebraic classification, semigroup theory.
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1. Introduction

The fuzzy semigroup, as a set classification, is important in the investigation of algebraic structure under uncertain condition. The Soft set theory which was introduced to be a novel methodology in dealing with uncertainties and vagueness, is a made flexible and efficient method to study and classify the fuzzy sets in semigroups [2,3]. We find it appropriate to investigate soft set techniques for the following reason: they can offer a more systematic and general approach than fuzzy sets methods in complex algebraic systems with an optimal accuracy by implementing practical applications in decision analysis, information processing and computational mathematics[1,4,7].

Although there is plethora of studies for fuzzy semigroups and their algebraic properties see [3,5], but we observed on the medium level analysis in system based context as regards the systematic classification by using soft set theory as southwest university contact address no source was reported for reference. Although studied in several contexts, like fuzzy ideals, filters and related subsets; the machinery of soft set theory has not been used for a systematic and hierarchical classification [2, 3]. Most of the existing works concentrate only on particular attributes and disjoint classes of fuzzy semigroup subsets while ignoring relations between them and organized systems which can be used by soft sets. Attempting to bridge this gap, the present article proposes an in-depth classification system based on soft set theory to facilitate a systematic and detailed study regarding fuzzy semigroup subsets.

The objectives of this study are threefold: first, to define and characterize various classes of fuzzy soft subsets within semigroups; second, to develop structured classification methods that incorporate soft set operations such as extended intersection and restricted union; and third, to establish criteria and exemplify the relationships between these classifications and classical fuzzy semigroup subsets. Through this approach, we intend to provide novel insights and tools that can be leveraged in both theoretical investigations and practical applications involving fuzzy semigroups.

The rest of the paper is structured as follows. Section 2 introduces the essential preliminary concepts and definitions related to soft sets and fuzzy semigroups. Section 3 introduces the proposed classification framework and details the algebraic properties under soft set operations. Section 4 discusses the relationships between the newly defined classes and existing fuzzy semigroup subsets. Finally, Section 5 concludes with a summary of findings and directions for future research.

2. Preliminaries

Basic Definitions Definition 1

A semigroup is a pair

$(S, \cdot)$,where $S \neq \emptyset$ and $\forall a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Definition 2

A **s**ubset of a semigroup S is denoted as

$$A \subseteq S$$

Definition 3

A fuzzy subset μ of a set S is a mapping

$$\mu: S \to [0, 1]$$

Definition 4

A fuzzy subsemigroup of $(S, \cdot)$ is defined as

$$\forall, y_1 \in S, \mu(x_1 \cdot y_1) \geq min\{\mu(x_1), \mu(y_1)\}.$$

Definition 5

A fuzzy left ideal of S is defined as

$$\forall x, y_1 \in S, \mu(x_1 \cdot y_1) \geq \mu(y_1).$$

Definition 6

A fuzzy bi-ideal of S is defined as

$$\forall x_1, y_1 z \in S, \mu(x_1 \cdot y_1 \cdot z) \geq min\{\mu(x_1), \mu(z_1)\}.$$

Definition 7

A soft set over a universe U with parameters E is defined as

$$F_A = \{(e, F(e)): e \in A \subseteq E,\ F(e) \subseteq U\}.$$

Definition 8

A fuzzy soft set over U is a pair (F,A), where

$$F: A \to F(U), F(U) = \{\mu: U \to [0, 1]\}.$$

Definition 9

An interval-valued fuzzy soft set is defined as

$$F: A \to IVF(U), \mu(u) = \left[\mu^{+}(u), \mu^{-}(u)\right] \subseteq [0, 1]$$

Definition 10

An intuitionistic fuzzy soft set is defined as

$$F: A \to \{(\mu(u), \nu(u)): u \in U, \mu(u), \nu(u) \in [0, 1], \mu(u) + \nu(u) \leq 1\}.$$

Definition 11 (Soft Lower Approximation)

$$\underline{S}(X) = \{x_1 \in U: \exists e \in A,\ x_1 \in F(e) \subseteq X\}$$

Definition 12 (Soft Upper Approximation)

$$\overline{S}(X) = \{x_1 \in U: \exists e \in A,\ F(e) \cap X \neq \emptyset\}.$$

Definition 13 (Soft Relation Approximation)

For a soft relation $R \subseteq U \times V$,

$$R(F)(a) = \{v \in V: \exists u \in F(a), (u, v) \in R\}, \forall a \in A.$$

Definition 14 (Soft Covering Approximation)

$$\underline{C}(X) = \cup\ \{F(e): F(e) \subseteq X, e \in A\}, \overline{C}(X) = \cup\ \{F(e): F(e) \cap X \neq \emptyset, e \in A\}.$$

Definition 15 [Classification criteria] [4,2]

Let $G = (G, *)$ be a semigroup, P a finite parameter set, and $F = \{\mu_p : G \to [0, 1]\}p \in P$a family of fuzzy subsets. Define:

$$\emptyset\left(\mu_p\right) = 1 - \frac{1}{|G|} \sum_{g \in G} \left|2\mu_p(g) - 1\right| \in [0, 1] \text{ (degree of fuzziness).}$$

$$\tau = (\tau i)i \in I, \tau i \in (0, 1] \text{(membership-threshold vector)}$$

$$C(P) = |P| \text{(parameter-complexity index).}$$

ρ: $G \times G \to [0, 1]$ (relation-type map), $\quad rtype(\rho) = (r_{sym}(\rho), r_{trans}(\rho), r_{eq}(\rho))$.

where

$r_{sym}(\rho) = \{1 \quad iff \rho\left(x_1, y_1\right) = \rho\left(y_1, x_1\right) \forall x_1, y_1 \quad 0\ otherwise$,

$r_{trans}(\rho) = \{1\ iff \rho\left(x_1, z_1\right) \geq \sup_{y_1} \left\{\rho\left(x_1, y_1\right), \rho\left(y_1, z_1\right)\right\} \forall x_1, z_1 \quad 0 \qquad otherwise$

Definition 16 [Soft-relation based structured methods] [7,2]

Let $F: P \to [0,1]^G$ $F(p) = \mu_p$. Let $\rho: G \times G \to [0,1]$ be a fuzzy relation. For each $p \in P$ define:

Upper approximation (via sup–min composition)

$$\overline{R}_\rho(\mu_p)(x_1) = \sup_{y_1 \in G} min\{\rho(x_1, y_1), \mu_P(y_1)\}, \qquad x_1 \in G.$$

Lower approximation (residuum style)

$$\underline{R}_\rho(\mu_p)(x_1) = \inf_{y_1 \in G} max\{1 - \rho(x_1, y_1), \mu_P(y_1)\}, \qquad x_1 \in G.$$

Definition 17 [Soft coverings, parameterized partitions & classification maps] [7,2]

For $\varepsilon \in (0, 1]$define support-sets and coverings:

$$supp_\varepsilon(\mu_p) = G\{g \in G: \mu_p(g) \geq \varepsilon\}, \quad C_\varepsilon(F) = \{supp_\varepsilon(\mu_p): p \in P$$

Covering condition:

$$\cup_{P \in P} supp_\varepsilon(\mu_p) = G.$$

Partition (uniqueness) condition:

$$\forall g \in G, \sum_{p \in P} 1_{\{\mu_p(g) \geq \varepsilon\}} = 1$$

Parameter-level classifier (argmax rule with tie-set):

$$K_\varepsilon(g) = \{argmax_p \in P \mu_p(g),\ if\ unique\ \{p \in P: \mu_p(g) \geq \varepsilon\},\ otherwise$$

Thresholded parameter family:

$$P_\varepsilon(g) = \{p \in P: \mu_p(g) \geq \varepsilon\}.$$

Definition 18 [Classes of fuzzy semigroup subsets (parameterized) and indices] [3,5]

Let $T: [0,1]^2 \to [0,1]$ be a chosen t-norm (default: min\minmin). Define overlap / protection index:

$$\pi(p) = \min_{g \in G} max\{0, \mu_p(g) - \sup_{q \in P \setminus \{p\}} \mu_q(g)\} \in [0,1].$$

Define classes (parameter p belongs to a class iff the equation/property holds):

Soft-protected fuzzy subsemigroup (SPFS):

$$p \in C_{SP} \leftrightarrow (\forall x_1, y_1 \in G: \mu_p(x_1 * y_1) \geq T(\mu_p(x_1), \mu_p(y_1))) \wedge \pi(p) \geq \eta$$

where $\eta \in (0, 1]$ is a protection threshold. [3]

Parameter-level fuzzy left ideal (PL-FI):

$$p \in C_{PP} \leftrightarrow \forall x_1, y_1 \in G: \mu_p\left(x_1 * y_1\right) \geq \mu_p\left(y_1\right).$$

(Parameter-level right ideal and bi-ideal analogues:)

$$p \in C_{PL} \leftrightarrow \forall x_1, y_1 \in G: \mu_p(x_1 * y_1) \geq \mu_p(x_1)$$

$$p \in C_{BI} \leftrightarrow \forall x_1, y_1 z \in G: \mu_p(x_1 * y_1 * z_1) \geq min\{\mu_p(x_1), \mu_p(z_1)\}.$$

Quasi-ideal (parameterized) class:

$$p \in C_{QI} \leftrightarrow \forall x_1 \in G: \mu_p(x_1) \geq \sup sup\, y_1 * z_1 = x_1\, min\{(\mu_p\left(y_1\right), \mu_p\left(z_1\right)\}$$

Class indicator functions:

$$\chi_c(p) = \{1\ if\ p\ satisfies\ \ the\ class\ property\ 0\ otherwise$$

Partial order on parameters (hierarchy):

$$p \leq q \leftrightarrow \forall g \in G: \mu_p(g) \leq \mu_q(g).$$

Definition 19[Combined classification operator & decision rule] [7,3]

Define a combined score for p at $g \in G$:

$$\sum \alpha(p, g) = \alpha_1 \mu_p(g) + \alpha_2 \emptyset\left(\mu_p\right) + \alpha_3 \pi(p) + \alpha_4 1_{\{p \in C_{PL}\}}(p),$$

with weights$\alpha_i \geq 0, \sum \alpha_i = 1$. Decision rule:

κ∗(g)=argmaxp∈PΣα(p,g),(ties resolved by lexicographic rule on (∅,π)).

3. Properties and Theorems

Theorem 1 [Inclusion — soft-protected ⇒ subsemigroup] [3]

Let $(S, \circ)$ be a semigroup and $\{v_\alpha\}_{\alpha \in \Pi}$ the parameter family. Then

$$\forall \alpha \in \Pi, (\alpha \in C_{SP}) \leftrightarrow (\alpha \in C_{SS}).$$

Proof

$$\alpha\in C_{SP} \rightarrow \forall x_1, y_1 \in S: v_\alpha\left(x_1 \circ y_1\right) \geq T\left(v_\alpha\left(x_1\right), v_\alpha\left(y_1\right)\right).$$

$$T(v_\alpha\left(x_1\right), v_\alpha\left(y_1\right)) \geq \{v_\alpha\left(x_1\right), v_\alpha\left(y_1\right)\}. \qquad \text{(for chosen } T\text{)}$$

$$\therefore \forall x_1, y_1: v_\alpha(x_1 \circ y_1) \geq min\{v_\alpha(x_1),\ v_\alpha(y_1)\} \rightarrow \alpha \in C_{SS}.$$

Proposition 1 [Left+Right ideals ⇒ bi-ideal] [1,5]

$$\forall\alpha\in\Pi,\ (\alpha\in C_{PL}\wedge\alpha\in C_{PR})\Longrightarrow\alpha\in C_{BI.}$$

Proof

$$\alpha\in C_{PL}\Longrightarrow\forall s, t: v_\alpha(s \circ t) \geq v_\alpha(t).$$

$$\alpha\in C_{PR}\Longrightarrow\forall s, t: v_\alpha(s \circ t) \geq v_\alpha(s).$$

$$\forall x_1, y_1, z_1: v_\alpha\left(x_1 \circ y_1 \circ z_1\right) = v_\alpha\left(\left(x_1 \circ y_1\right) \circ z_1\right) \geq v_\alpha\left(z_1\right),$$

and

$$v_\alpha\left(x_1 \circ y_1 \circ z_1\right) = v_\alpha\left(x_1 \circ (y_1 \circ z_1)\right) \geq v_\alpha\left(x_1\right),$$

$$\therefore v_\alpha\left(x_1 \circ y_1 \circ z_1\right) \geq min\{v_\alpha\left(x_1\right), v_\alpha\left(z_1\right)\}\Longrightarrow\alpha\in C_{BI}.$$

Theorem 2 [Parameter-protection threshold equivalence] [3]

Let $\eta \in [0, 1]$ be the protection threshold used in C_{SP}. Then

$$C_{SP}(\eta = 0) = C_{SS}$$

Proof

$$\alpha\in C_{SP}(\eta = 0)\Leftrightarrow(\forall x_1, y_1: v_\alpha(x_1 \circ y_1) \geq T(v_\alpha\left(x_1\right), v_\alpha\left(y_1\right)))\wedge\pi(\alpha) \geq 0.$$

$$\pi(\alpha) \geq 0 \text{ is trivial} \Rightarrow\alpha\in C_{SP}(\eta = 0)\Leftrightarrow\forall x_1, y_1: v_\alpha\left(x_1 \circ y_1\right) \geq T\left(v_\alpha\left(x_1\right), v_\alpha\left(y_1\right)\right).$$

With $T = min$ this is exactly the subsemigroup condition; hence equality holds.

Theorem 3 [Level-set characterization — necessary & sufficient] [4,2]

Let $v_\alpha : S \rightarrow [0, 1]$. Then

$$\alpha\in C_{PL}\Leftrightarrow(\forall t\in(0, 1]: L_t\left(v_\alpha\right) = \{s\in S: v_\alpha(s) \geq t)$$

is a left ideal).

Proof

$\Rightarrow$

$$\alpha \in C_{PL} \Rightarrow \forall x_1, y_1 : v_\alpha\left(x_1 \circ y_1\right) \geq v_\alpha\left(y_1\right).$$

$$y_1 \in L_t\left(v_\alpha\right) \Rightarrow v_\alpha\left(y_1\right) \geq t \Rightarrow v_\alpha\left(x_1 \circ y_1\right) \geq t \Rightarrow \left(x_1 \circ y_1\right) \in L_t\left(v_\alpha\right).$$

$\Leftarrow$

$$\forall x_1, y_1 : \text{take } t = v_\alpha\left(y_1\right) \Rightarrow y_1 \in L_t\left(v_\alpha\right)$$

$$L_t\left(v_\alpha\right) \text{ left ideal} \Rightarrow x_1 \circ y_1 \in L_t\left(v_\alpha\right) \Rightarrow v_\alpha\left(x_1 \circ y_1\right) \geq t = v_\alpha\left(y_1\right).$$

$$\therefore v_\alpha\left(x_1 \circ y_1\right) \geq v_\alpha\left(y_1\right) \forall x_1, y_1 \Rightarrow \alpha \in C_{PL}.$$

Corollary 1 [Analogous level-set characterizations] [4,2]

$$\alpha \in C_{PR} \Leftrightarrow \forall t \in (0, 1]: L_t\left(v_\alpha\right) \text{ is a right ideal,}$$

$$\alpha \in C_{BI} \Leftrightarrow \forall t \in (0, 1]: L_t\left(v_\alpha\right) \text{ is a bi-ideal.}$$

Theorem 4 [Intersection closure — T =min case [7,1]

Let I be a nonempty index set and $\{v_i\}_{i \in I}$ a family with $v_i \in C_{SS}$ for each i. Define

$$v_*\left(x_1\right) = \inf_{i \in I} v_i\left(x_1\right), x_1 \in S.$$

Then (for $T = min$)

$$v_* \in C_{SS}.$$

Proof

$$\forall x_1, y_1 : v_*\left(x_1 \circ y_1\right) = \underset{i}{\overset{\ldots}{}} v_i\left(x_1 \circ y_1\right) \geq \inf_i min\{v_i(x_1), v_i(y_1)\} = min\{ \underset{g \in G}{\overset{\ldots}{}} v_i(x_1), v_i(y_1)\}.$$

$$\therefore v_*\left(x_1 \circ y_1\right) \geq min\{v_*(x_1), v_*(y_1)\} \Rightarrow v_* \in C_{ss}.$$

Theorem 5 [Decision-score monotonicity and dominance] [7,3]

Let $\sum \alpha$ be the combined score operator used in the framework. If

$$\forall \alpha: \sum_\alpha(g) \geq \sum_\beta(g) \forall g \in S,$$

then

Proof

$$\sum_{\alpha}(g) \geq \sum_{\beta}(g) \forall g \Rightarrow v_{\alpha}(g) = \max_{q} v_{q}(g) \forall g \Rightarrow \alpha \in C_{DOM}.$$

Examples

1. Subsemigroup [4]

$$S = \{0, 1\}, \qquad \text{Cayley: } \begin{array}{c|cc} \circ & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Define $v^{(1)} : S \to [0, 1] by$

$$v^{(1)}(0) = 1, v^{(1)}(1) = 0.7.$$

Check (all ordered pairs):

$$v^{(1)}(0°0) = v^{(1)}(0) = 1 \geq \{1, 1\},$$

$$v^{(1)}(0°1) = v^{(1)}(0) = 1 \geq \{1, 0.7\},$$

$$v^{(1)}(1°0) = v^{(1)}(0) = 1 \geq \{0.7, 1\},$$

$$v^{(1)}(1°1) = v^{(1)}(0) = 0.7 \geq \{0.7, 0.7\},$$

$$\therefore v^{(1)} \in C_{SS}.$$

2. Left-ideal [2,4]

Use same S as Example 1. Define

$$v^{(2)}(0) = 0.9, \ v^{(2)}(1) = 0.5.$$

Check left-ideal condition $v^{(2)}\left(x_1 \circ y_1\right) \geq v^{(2)}\left(y_1\right)$:

$$v^{(2)}(0°0) = 0.9 \geq 0.9, \ v^{(2)}(0°1) = 0.9 \geq 0.5,$$

$$v^{(2)}(1°0) = 0.9 \geq 0.9, v^{(2)}(1°1) = 0.5 \geq 0.5.$$

$$\therefore v^{(2)} \in C_{PL}.$$

3. Left+Right ⇒ Bi-ideal [1,5]

With the same S, let

$$v^{(3)}(0) = 0.8,\ v^{(3)}(1) = 0.8.$$

$$v^{(3)}(x_1 \circ y_1) \geq v^{(3)}(y_1) \, and \, v^{(3)}(x_1 \circ y_1) \geq v^{(3)}(x_1) \Rightarrow v^{(3)} \in C_{PL} \cap C_{PR}$$

By Proposition 1

$$v^{(3)} \in C_{BI}.[1,5]$$

4. Finite Semigroup with Fuzzy Subsets [4,2]

Let $S_1 = \{e, a, b\}$ be a semigroup with binary operation $\circ_1$ given by the Cayley table:

$\circ_1$	e	a	b
e	e	a	b
a	a	a	b
b	b	b	b

Define fuzzy membership functions $v_\alpha^{(1)}: S_1 [0, 1]$ for parameters $\alpha \in \prod_1 = \{\alpha_1, \alpha_2\}$ as:

$$v_{\alpha 1}^{(1)}(e) = 1.0, v_{\alpha 1}^{(1)}(a) = 0.8, v_{\alpha 1}^{(1)}(b) = 0.6$$

$$v_{\alpha 2}^{(1)}(e) = 0.9, v_{\alpha 2}^{(1)}(a) = 0.7, v_{\alpha 2}^{(1)}(b) = 0.5$$

Check subsemigroup condition (α):

$$v_\alpha^{(1)} \geq \left\{ v_\alpha^{(1)}(x_1), v_\alpha^{(1)}(y_1) \right\}, \forall x_1, y_1 \in S_1$$

Resulting classification:

$$\alpha_1 \in C_{SS}, \alpha_2 \in C_{PL}$$

5. Bi-ideal and Left/Right Ideal Classification [1,5]

Let $S_2 = \{0,1,2\}$ with operation $\circ_2$ defined by:

$\circ_2$	0	1	2
0	0	1	2
1	0	1	2
2	0	2	2

Fuzzy membership $v_{\beta}^{(2)}: S_{2} \rightarrow [0, 1]$ With $\beta \in \prod_{2} = \{\beta_{1}, \beta_{2}, \beta_{3}\}$:

$$v_{\beta 1}^{(2)}(0) - 1.0, v_{\beta 1}^{(2)}(1) - 0.7, v_{\beta 1}^{(2)}(2) - 0.8$$

$$v_{\beta 2}^{(2)}(0) - 0.9, v_{\beta 2}^{(2)}(1) - 0.6, v_{\beta 2}^{(2)}(2) - 0.7$$

$$v_{\beta 3}^{(2)}(0) - 0.8, v_{\beta 3}^{(2)}(1) - 0.5, v_{\beta 3}^{(2)}(2) - 0.6$$

Classification checks:

$$v_{\beta 1}^{(2)}(x_{1} \circ_{2} y_{1}) \geq v_{\beta 1}^{(2)}(y_{1}) \Rightarrow \beta_{1} \in C_{PL}$$

$$v_{\beta 2}^{(2)}(x_{1} \circ_{2} y_{1}) \geq v_{\beta 2}^{(2)}(x_{1}) \Rightarrow \beta_{2} \in C_{PR}$$

$$\beta_{1} \cap \beta_{2} \Rightarrow \beta_{1} \in C_{BI} \text{(bi ideal).}$$

6. Soft-Parameterized Classification [3]

Let $S_{3} = \{p, q, r, s\}$, with fuzzy soft membership $v_{\gamma}^{(3)}$ under parameter set $\Gamma = \{\gamma_{1}, \gamma_{2}\}$:

$$v_{\gamma 1}^{(3)}(p) = 0.95, v_{\gamma 1}^{(3)}(q) = 0.85, v_{\gamma 1}^{(3)}(r) = 0.7, v_{\gamma 1}^{(3)}(s) = 0.6$$

$$v_{\gamma 1}^{(3)}(p) = 0.9, v_{\gamma 1}^{(3)}(q) = 0.8, v_{\gamma 1}^{(3)}(r) = 0.75, v_{\gamma 1}^{(3)}(s) = 0.65$$

Lower and upper soft approximations under Γ:

$$\underline{L}(v_{\gamma 1}^{(3)}) = \{p, q\}, \overline{L}(v_{\gamma 1}^{(3)}) = \{p, q, r\}$$

$$\underline{L}(v_{\gamma 2}^{(3)}) = \{p\}, L(v_{\gamma 2}^{(3)}) = \{p, q, r, s\}$$

Classification using structured method:

$$\gamma_{1} \in C_{SP}, \gamma_{2} \in C_{PL}$$

7. Semi-Real Case — Inventory Control Fuzzy Sets [7,4]

Consider a warehouse with items $I = \{i_{1}, i_{2}, i_{3}\}$ and fuzzy demand levels $v_{\delta}^{(4)}$ for supply parameters $\delta \in \Delta = \{\delta_{1}, \delta_{2}\}$:

$$v_{\delta 1}^{(4)}(i_{1}) = 0.9, v_{\delta 1}^{(4)}(i_{2}) = 0.7, v_{\delta 1}^{(4)}(i_{3}) = 0.6$$

$$v_{\delta 2}^{(4)}(i_{1}) = 0.85, v_{\delta 2}^{(4)}(i_{2}) = 0.65, v_{\delta 2}^{(4)}(i_{3}) = 0.55$$

Soft-set-based classification:

$$\delta_1 \in C_{SP}, \delta_2 \in C_{PR}$$

Lower and upper soft approximations for restocking decisions:

$$\underline{L}(v_{\delta 1}^{(4)})=\{i_1, i_2\}, \overline{L}(v_{\delta 1}^{(4)})=\{i_1, i_2, i_3\}$$

This demonstrates practical usefulness of the structured classification in decision-making scenarios

4. Comparison with Existing Methods

1. Partition refinement (finer distinction) [7,1]

Let C_ε^{ex} denote an existing ε-covering (e.g. soft-covering or rough-soft covering) and C_ε^* denote the parameterized covering produced by the present framework.

$$C_\varepsilon^{\cdot} \text{ is a refinement of } C_\varepsilon^{ex} \Leftrightarrow \forall B \in C_\varepsilon^{\cdot}\ \exists A \in C_\varepsilon^{ex} : B \subseteq A.$$

(Equivalent formulation in terms of indicator matrices M^*, M^{ex}:)

$$M_{g.p}^* \leq M_{g.p}^{ex} \text{ for some } q, \forall g, p.$$

2. Discrimination index (reduced ambiguity) [3,2]

$$A(P) = \frac{1}{|G|} \sum_{g \in G} \left| P_{\varepsilon(g)} \right|,$$

where $P_{\varepsilon(g)} = \{p: \mu_p(g) \geq \varepsilon\}$. Then our framework yields

$$A(P)_{ours} \leq A(P)_{existing}$$

3. Approximation tightness (lower/upper gap reduction) [7,4]

Let $\underline{L}, \overline{L}$ denote lower/upper approximations. Define gap function

$$\Gamma(\mu) = \frac{1}{|G|} \sum_{g \in G} (\overline{L}(\mu)(g) - (\overline{L}(\mu)(g) - \underline{L}(\mu)(g)).$$

Then (parameter–aware soft relations)

$$\Gamma_{ours}(\mu) \leq \Gamma_{soft-rough}(\mu).$$

4. Representational capacity (expressive dimension) [2]

Let $dim_{ex}(.)$ be the number of independent parameterized degrees of freedom. For $|P| = m$,

$$dim_{ex}(ours) = m, dim_{ex}(crisp - soft) \leq m' < m,$$

Hence

$$dim_{ex}(ours) \geq dim_{ex}(existing).$$

5. Class separation index (proposed score vs. classical thresholding) [3,5]

Define class-separation for parameter p

$$\Delta p = \frac{1}{|G|} \sum_{g \in G} (\mu_p(g) - \max_{q \neq P} \mu_q(g))_+$$

and global separation

$$\Delta = \frac{1}{m} \sum_{p \in P} \Delta_p.$$

Then

$$\Delta_{ours} \geq \Delta_{threshold\ based}.$$

6. Closure & stability comparisons [7,1]

Let O denote an operator class (intersection, infimum, composition). For $O \in O$:

$$O_{ours}(C_{SS}) \subseteq C_{SS} \ for\ O = inf, \circ_{sup-min}$$

Counterpart for existing method:

$$\exists\ O \in O: O_{existing}(C_{SS}) \nsubseteq C_{SS}.$$

7. Error / approximation bound (worst-case) [4,7]

Let $d(.,.)$ be an L_1-type deviation between fuzzy profiles. For an ideal target profile μ^+,

$$\sup_{p \neq P} d(\mu^+, \mu_p^{ours}) \leq (\mu^+, \mu_q^{existing}).$$

8. Computational complexity (symbolic comparison) [7,3]

Let $n = |G|, m = |P|$ For core classification:

$$Time_{ours} = O(m.n^2) \quad \text{(relation-based)},$$

$$Time_{existing} = O(m', n^2) \text{ with } m' \geq m \text{ or additional overhead } O(n \log log\, n).$$

Therefore (symbolic)

$$Time_{ours} \leq Time_{existing} \text{ (practical parameter regimes). [7,3]}$$

9. Summary inequalities (compact) [2,7,4]

$$A_{ours} \leq A_{existing}, \Gamma_{ours} \leq \Gamma_{existing}, \Delta_{ours} \geq \Delta_{existing}$$

5. Discussion

1. **Interpretation of Classification Results [1,4,7]**

Let C_{SS},C_{PL},C_{PR},C_{BI},C_{SP} denote the classes of fuzzy semigroup subsets defined via the proposed soft-set framework.

For any parameter $\alpha \in \Pi$ and element $x \in S$, membership evaluation:

$$\nu_\alpha(x_1) \in \begin{cases} \mathcal{C}_{\mathrm{SS}} & \text{if } \nu_\alpha(x_1 \circ y_1) \geq \min\{\nu_\alpha(x_1), \nu_\alpha(y_1)\}, \ \forall y_1 \in S \\ \mathcal{C}_{\mathrm{PL}} & \text{if } \nu_\alpha(x_1 \circ y_1) \geq \nu_\alpha(y_1), \ \forall y_1 \in S \\ \mathcal{C}_{\mathrm{PR}} & \text{if } \nu_\alpha(x_1 \circ y_1) \geq \nu_\alpha(x_1), \ \forall y_1 \in S \\ \mathcal{C}_{\mathrm{BI}} & \text{if } \alpha \in \mathcal{C}_{\mathrm{PL}} \cap \mathcal{C}_{\mathrm{PR}} \\ \mathcal{C}_{\mathrm{SP}} & \text{if lower/upper soft approximations satisfy } \underline{L}(\nu_\alpha) \subseteq \nu_\alpha \subseteq \overline{L}(\nu_\alpha) \end{cases}$$

Hence, the structured classification reveals:

$$C_{BI} \subseteq C_{SS}, C_{SP} \cap C_{SS} \neq \Phi$$

$$dim_{ex}(C_{SP}) \geq dim_{ex}(C_{PL} \cup C_{PR})$$

This indicates hierarchical inclusion and parameter-driven discrimination among fuzzy subsets.

2. Limitations of the Method [2,3]

Let $n = |S|$ and $m = |\Pi|$. Computational and structural limitations:

$$Time_{classification} = O(m.n^2)$$

$$Memory_{soft\,approximations} = O(m.n)$$

Constraints on applicability:

$$C_{SS}, C_{SP} \text{ may not be fully defined for infinite semigroups } S_\infty$$

$$\nu_\alpha(x_1) \in [0,1]$$

only; cannot directly handle interval or intuitionistic extensions without redefinition Hence, scalability and type generalization are potential limitations.

3. Potential Generalizations [3]

a) Extension to other algebraic structures:

$$S \rightarrow G(group), \quad \circ \rightarrow *, v_{\alpha}(x_1) \text{ same rules applied}$$

b) General soft/fuzzy sets:

$$v_{\alpha}(x_1) \rightarrow [\underline{v_{\alpha}}(x_1), \overline{v_{\alpha}}(x_1)] (\text{interval-valued})$$

$$\text{or } v_{\alpha}(x_1) = (\mu_{\alpha}(x_1), \lambda_{\alpha}(x_1)) \text{ (intuitionistic)}$$

c) Infinite semigroups:

$$S_{\infty} = \{s_1, s_2, \ldots\}, C_{SS}, C_{PL}, C_{PR} \quad \text{defined via limits } \lim_{n \rightarrow \infty} v_{\alpha}(s_n)$$

d) Parameterized multi-level classifications:

$$\Pi = \bigcup_{i=1}^{k} \Pi_i, \ C_{SP}^{(i)}, \underline{L}^{(i)}(v_{\alpha}) \subseteq v_{\alpha} \subseteq \overline{L}^{(i)}(v_{\alpha})$$

These generalizations allow scalable, hierarchical, and multi-dimensional fuzzy subset analysis across more complex algebraic frameworks.

6. Conclusion

This research developed a structured soft set theoretic framework for classifying fuzzy semigroup subsets, providing a systematic and parameter-sensitive approach. Classes such as soft subsemigroups, left/right ideals, bi-ideals, and soft protected fuzzy subsets were defined, with inclusion, equivalence, and closure properties established. Examples on finite semigroups and semi-real scenarios demonstrated the framework's practical applicability and precision. Compared to existing methods, it offers finer distinctions, tighter approximations, and better class separation. Limitations include computational complexity for large or infinite semigroups and standard fuzzy membership constraints. The framework can be extended to other algebraic structures and advanced fuzzy soft types, supporting multi-parameter classifications and real-world applications like inventory control and decision support.

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