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A Simple Geometric Theorem with a Constructible Configuration Whose Truthfulness Depends on the Base Field Considered

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Abstract- It is well-known that the truthfulness of some theorems in plane geometry depends on the base field considered. We shall show a simple example of geometric theorem, whose corresponding geometric configuration is rule-and-compass constructible in the real case, that is always true in the real case and that is "almost always" false in the complex case (more precisely, it is true only in a non-degenerated algebraic variety). We believe this is an eye-catching example illustrating how the results from explorations carried out with a Dynamic Geometry System or from computations carried out using a Computer Algebra System should be carefully analyzed.

1 Introduction

Let us begin with two elementary introductory examples of algebraic systems that illustrate how different the real and complex solutions of a geometric problem can be (this section can be skipped by any reader with notions of elementary analytic geometry).

Example 1: Let su consider the following algebraic system in $\mathbb{R}[x, y]$:

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ x^2 + y^2 - 4 = 0 \end{cases}$$

If we plot the zeroes of the two equations, that is, the so called *algebraic variety* of the *ideal* generated by the polynomials in the left hand side of the equations, two concentric circumferences are obtained (see Figure 1). Obviously, the two circumferences do not intersect, so the system has no real solution (i.e., no solution in $\mathbb{R}^2$).

The example is simple and it can be easily solved by hand:

$$\left.\begin{array}{l} x^2 + y^2 - 1 = 0 \\ x^2 + y^2 - 4 = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow (x^2 + y^2 - 1) - (x^2 + y^2 - 4) = 0 \Rightarrow 3 = 0$$

so the system has no complex solutions either (i.e., it has no solutions if the polynomials are considered in $\mathbb{C}[x, y]$ and we look for solutions in $\mathbb{C}^2$).

Example 2: If we consider the algebraic system in

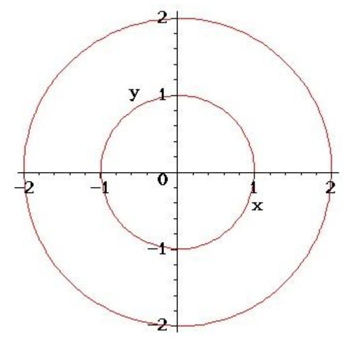

Figure 1: The algebraic system corresponding to Example 1 has neither real nor complex solutions

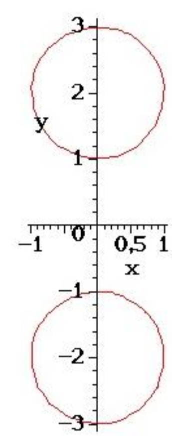

Figure 2: The algebraic system corresponding to Example 2 has no real solutions but has complex solutions

$\mathbb{R}[x,y]$:

$$\begin{cases} x^2+(y-2)^2-1^2=0 \\ x^2+(y+2)^2-1^2=0 \end{cases}$$

and we plot the zeroes of the two equations, two symmetric circles are obtained (Figure 2). Obviously, they do not intersect, so the system has no real solutions.

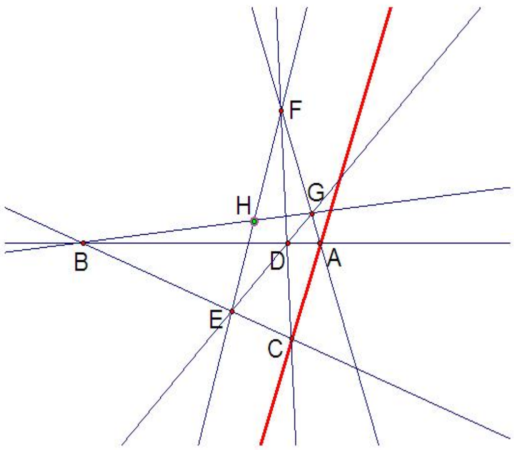

Figure 3: Trying to find a counterexample of MacLane 8_3 Theorem with the DGS *GSP*

But if we try to solve this system by hand:

$$\left.\begin{array}{l} x^2+(y-2)^2-1^2=0 \\ x^2+(y+2)^2-1^2=0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow (x^2+(y+2)^2-1^2)-(x^2+(y-2)^2-1^2)=0 \Rightarrow$$

$$\Rightarrow (y+2)^2-(y-2)^2=0 \Rightarrow 4y=0 \Rightarrow y=0$$

and substituting this value in any of the two equations:

$$x^2+4-1=0 \Rightarrow x=\pm\sqrt{3}\cdot I$$

It is straightforward to check that this two candidates are both solutions, so the system has two complex solutions: $(\pm\sqrt{3}\cdot I, 0)$.

2 MacLane 8_3 Theorem

It is well-known that the truthfulness of some theorems in plane geometry depends on the base field considered.

A well known tricky one is *MacLane 8_3 Theorem* [1]. It is extensively treated in excellent papers regarding automated theorem proving like [2, 3, 4, 5].

For instance in [3] it is enunciated as follows.

MacLane 8_3 Theorem: *Consider eight points $A, B, ..., H$ such that the following eight triples are collinear ABD, BCE, CDF, DEG, EFH, FGA, GHB, HAC. Then all eight points lie on a line. (All points A,B,...,H are supposed to be different from each other.)*

But, is this theorem true or false?

2.1 Exploring MacLane 8_3 with a Classic Dynamic Geometry System

A first attempt to try to show that MacLane 8_3 Theorem is false could be to try to find a configuration where the eight points were not aligned using a classic *Dynamic Geometry System* (DGS) like *The Geometer's Sketchpad* (*GSP*).

Nevertheless, all attempts with *GSP* fail. For instance, in Figure 3, only that H should lie on AC (the thick line) is missing. The dynamic exploration of the construction is very curious: H can get closer and closer to $\overline{BC}$ but, depending on which element we are dragging, either the movement of H suddenly changes its direction and H moves away from $\overline{BC}$, or the eight lines collapse into a single one.

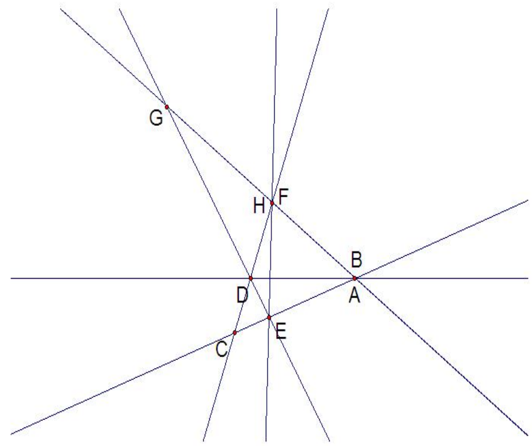

Figure 4: If the hypothesis condition that the 8 initial points are different from each other is excluded, then the thesis of MacLane 8_3 Theorem doesn't hold

If it was allowed that some of the initial points coincided, then it would be possible to find eight points verifying the alignment conditions. Two such configurations can be found in Figure 4 (two pairs of points coincide) and in Figure 5 (three pairs of points coincide).

2.2 Exploring MacLane 8_3 with the DGS *Geometry Expressions*

Geometry Expressions (*GE*) [6] is a recently released innovative DGS, that includes a small Computer Algebra System (CAS) and can communicate in a bidirectional way (using *MathML*) with the CAS *Maple* and *Mathematica*. Observe that the existing attempts to connect DGS with existing CAS, like *GDI* [7, 8, 9], *GEOTHER* [10], *paramGeo* [11]... are only able to export information from the DGS to the external CAS (*Geometry Expert* [12, 13] follows another philosophy, as it includes a built-in CAS).

Using *GE*, the same step as with *GSP* can be reached: only $H \in AC$ is missing (see Figure 6). As

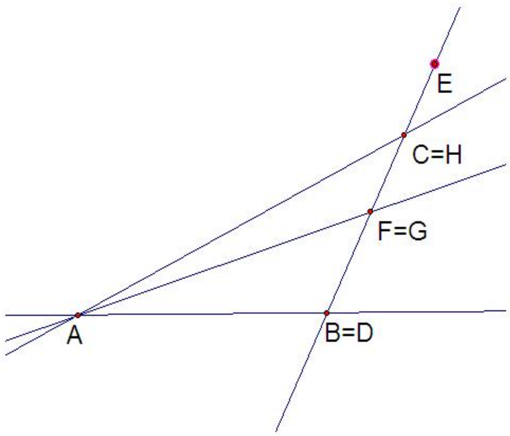

Figure 5: Another configuration where the hypothesis condition that the 8 initial points are different from each other is excluded and the thesis of MacLane 8_3 Theorem doesn't hold

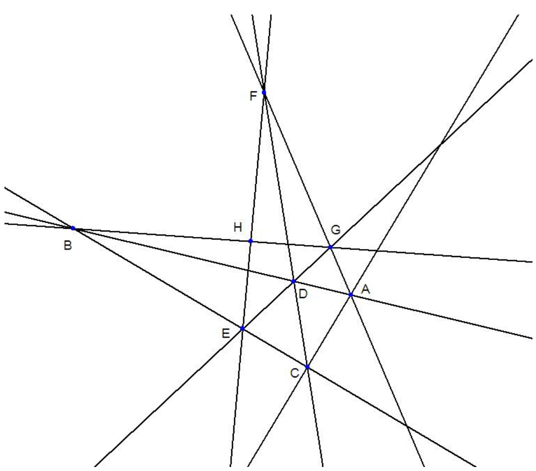

Figure 6: Dynamically trying to find a counterexample of MacLane 8_3 Theorem with the DGS *GE*

GE is a constraint-based DGS, we can then try to impose $H \in AC$ using *Constrain (Input)/Incident*, but an unsolvable *Constraint Conflict* is obtained (see Figure 7).

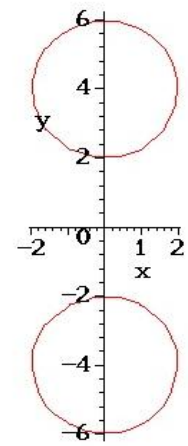

Figure 8: Configuration of Theorem 1 in $\mathbb{R}^2$ when $k = 1$

2.3 When Does MacLane 8_3 Theorem Hold?

As said above, this theorem is studied in detail, e.g., in [2, 3, 4, 5]. But how to rate its truthfulness is a controversial issue:

- in [3] it is said

  > "It holds in the real plane but fails in the complex one."

- in [5] it is treated in Example 4, that finishes saying

  > "... give a non-degenerated complex zero of $\mathbb{C}_{16}$, under which the eight points are not collinear. So the theorem is not true over the field of complex numbers."

  (in fact the non-degenerate complex zero of $\mathbb{C}_{16}$ is explicitly given)

- in [4], they claim

  > "... which implies [18] that the theorem holds over all components of the hypothesis variety where these variables (of number equal to the dimension of the variety) remain independent; this fact (for some of us) supports calling this a *generally true theorem*"

  Remarkably, the authors of [4] treat the problem using the new generation DGS *GDI*, that is able to communicate with the CAS *CoCoA* and *Mathematica* in order to perform algebraic computations.

Meanwhile,

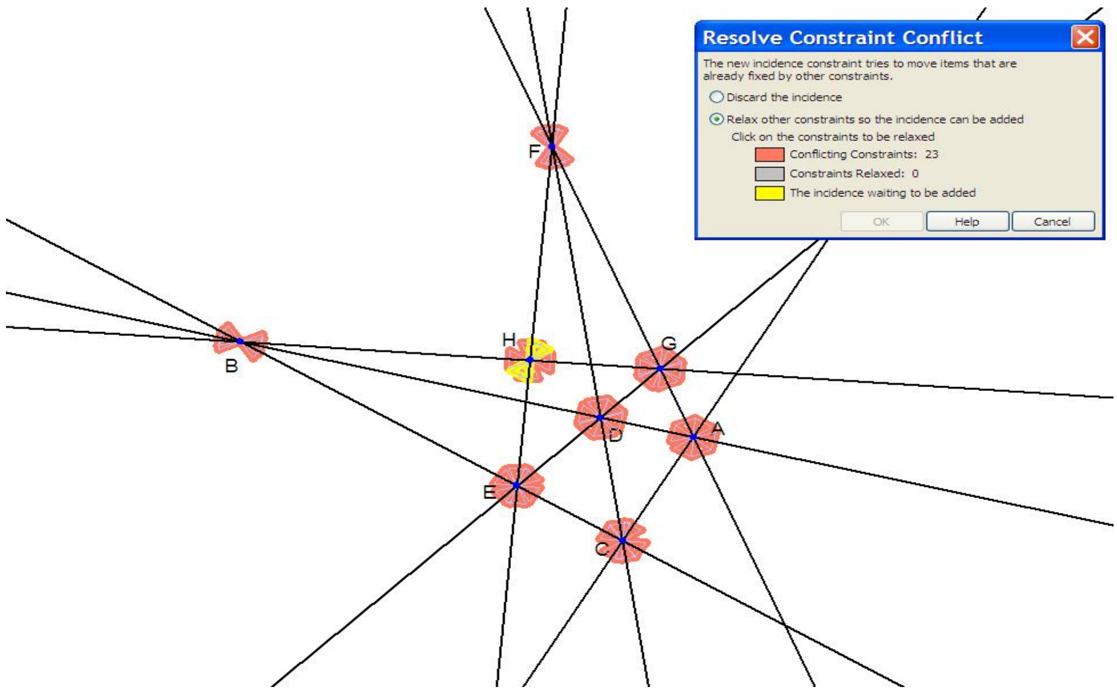

Figure 7: Trying to find a counterexample of MacLane 8_3 Theorem with *GE* by imposing a membership constraint

- in [2] is considered “A Weird Example”; and, interestingly, their conclusion is

> “The theorem stating that $P_1, ..., P_8$ are collinear is true on the reals. (...) The theorem is intuitively false on the complex field, since the component H_9 cannot be considered “degenerate”. But if we adhere to our definition of (algebraic) truth, the theorem should be considered true; an example of an obviously false theorem that is true.”

Essentially, they all reach the same conclusions, but the definition used of a “true” or “generally true” theorem is different (in the latter cases, the dimensions of the components of the varieties are analyzed, and, that the theorem holds in the highest dimension variety, is found).

The definition of “a generally true theorem” more commonly used is probably that in [14] (pages 48-49). A detailed definition of “algebraically true statement” can be found in [15] (Section 2). A specific discussion about grading the “truth of geometry theorems” can be found in [16].

3 A Curious Theorem

Now we wonder if it would be possible to find a simple theorem that verified the following assertions:

- it is rule-and-compass constructible in the real case,
- it is always true in the real case,
- it is “almost always” false in the complex case (it is true only in a non-degenerated algebraic variety).

The answer to the previous question is “yes”, and we have found a simple example: Theorem 1 verifies all these conditions, as will be shown below.

Theorem 1: Let $K = (k, 0)$ be a point on the x axis. Let c_1 and c_2 be two circumferences, both of them of radius $k^2 + 1$, and of centers $M_1 = (0, 2 \cdot (k^2 + 1))$ and $M_2 = (0, -2 \cdot (k^2 + 1))$, respectively. Then, for any value of k, the two circumferences c_1 and c_2 are either disjoint or do coincide.

Remark 1: According to the way Theorem 1 is stated, if it is considered as a theorem in $\mathbb{R}^2$, then $k \in \mathbb{R}$.

Remark 2: If Theorem 1 is considered as a theorem in $\mathbb{R}^2$, then the two circumferences are symmetrical w.r.t. the x axis.

Remark 3: The equations of the two circumferences described in Theorem 1 are:

$$\begin{cases} x^2 + (y - 2 \cdot (k^2 + 1))^2 - (k^2 + 1)^2 = 0 \\ x^2 + (y + 2 \cdot (k^2 + 1))^2 - (k^2 + 1)^2 = 0 \end{cases}$$

(we shall precise in each case if they will be considered as polynomials in $\mathbb{R}[x, y, k]$ or in $\mathbb{C}[x, y, k]$).

Example 3: (in $\mathbb{R}^2$) If $k = 1$, the circumferences c_1 and c_2 have radius 2 and centers $(4, 0)$ and $(-4, 0)$, respectively (see Figure 8), so they share no real points. Therefore, Theorem 1 holds in the reals at least when $k = 1$.

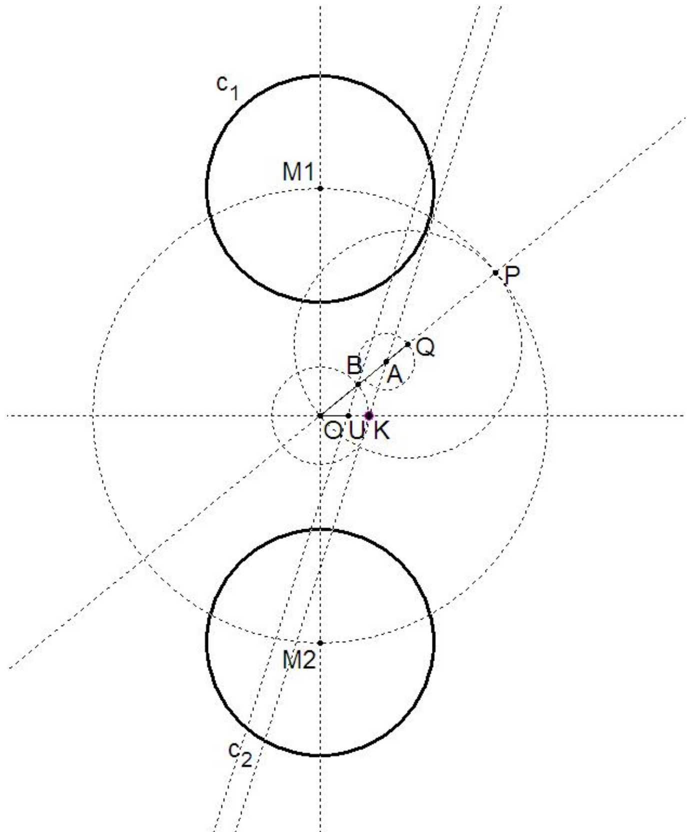

Figure 9: Configuration of Theorem 1 in $\mathbb{R}^2$ when $k = 1$

3.1 The Configuration of Theorem 1 is Rule-and-Compass Constructive in $\mathbb{R}^2$

A rule-and-compass constructive version (in $\mathbb{R}^2$) of the configuration of Theorem 1 could be (see Figure 9):

- let U be a point on the x^+ half-axis, such that $\overline{OU}$ is the unit segment,
- let K be a point on the x^+ half-axis, such that $length(\overline{OK}) = k$,
- draw a circumference of center O passing through K,
- let B be another point on this circumference,
- draw a parallel to UB through K and do intersect it with line OB,
- denote this intersection point by A; as triangles ΔOUB and ΔOKA are in Thales position, $length(\overline{OA}) = k^2$,
- draw a circumference of center A and radius $\overline{OU}$,
- intersect this circumference with line OA and denote by Q the further intersection point (from O); this way $length(\overline{OQ}) = length(\overline{OA}) + 1 = k^2 + 1$,
- draw a circumference of center Q passing through O,
- denote by P the intersection point of this circumference and line OQ that is not point O; this way $length(\overline{OP}) = 2 \cdot (k^2 + 1)$,
- draw points M_1 and M_2 on the y axes so that their distances to O are equal to $length(\overline{OP})$,
- denote by c_1 and c_2 the circumferences of radius $length(\overline{OQ})$ and centres M_1 and M_2, respectively.

3.2 Dynamically Exploring Theorem 1 in $\mathbb{R}^2$

We can intuitively treat the real case. Some DGS like *GSP* allow to produce animations and to make objects leave traces. Let us return to the construction of Figure 9. If we choose that c_1 and c_2 "leave trace" and we animate point K on the x^+ half-axis, Figure 10 is obtained. As the two shaded areas corresponding to the "traces" of c_1 and c_2 are disjoint, we experimentally confirm that the two circumferences never intersect in the real case.

3.3 Truthfulness of Theorem 1 in the Real Case

We can prove Theorem 1 in $\mathbb{R}^2$ just using synthetic techniques:

i) According to Remark 1, as we are stating Theorem 1 in $\mathbb{R}^2$, k is necessarily a real number and consequently $k^2 + 1 \geq 1$.

ii) The loci described in Theorem 1 are two circumferences of equal radii in $\mathbb{R}^2$ (as shown in Fig-

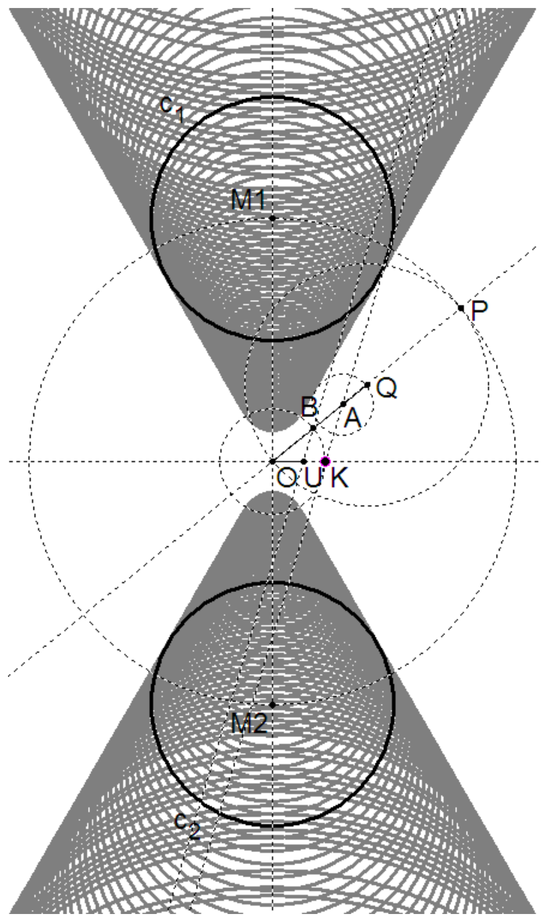

Figure 10: Animating the construction with *GSP*

ure 8). The distance between their centers is four times their radii, and their radii are equal to $k^2 + 1$. As shown in i), $k^2 + 1 \geq 1$, so the two circumferences are always disjoint, and consequently Theorem 1 holds in $\mathbb{R}^2$.

3.4 Truthfulness of Theorem 1 in the Complex Case

Le us consider Theorem 1 in $\mathbb{C}^2$. According to its enunciate, now $k \in \mathbb{C}$. The loci considered are again:

$$\begin{cases} x^2 + (y - 2 \cdot (k^2 + 1))^2 - (k^2 + 1)^2 = 0 \\ x^2 + (y + 2 \cdot (k^2 + 1))^2 - (k^2 + 1)^2 = 0 \end{cases}$$

Subtracting the first equation from the second one, we obtain:

$$(y+2\cdot(k^2+1))^2-(y-2\cdot(k^2+1))^2 = 0 \Rightarrow 8\cdot y\cdot(k^2+1) = 0$$

There are two possibilities:

a) If $k^2 + 1 = 0 \Leftrightarrow k = \pm I$ we have that $c_1 = c_2$ and that both circumferences have degenerated into the pair of lines:

$$y^2 + x^2 = 0 \Leftrightarrow y = \pm I \cdot x$$

so they would have infinite points in common. Therefore, Theorem 1 would be true.

b) If $k^2 + 1 \neq 0 \Leftrightarrow k \neq \pm I$, then the intersection must verify $y = 0$. For this value of y, we have that the system

$$\begin{cases} x^2 + (-2 \cdot (k^2 + 1))^2 - (k^2 + 1)^2 = 0 \\ x^2 + (2 \cdot (k^2 + 1))^2 - (k^2 + 1)^2 = 0 \end{cases}$$

is equivalent to the equation

$$x^2 + (-2 \cdot (k^2 + 1))^2 - (k^2 + 1)^2 = 0 \Leftrightarrow$$

$$\Leftrightarrow x^2 + 3 \cdot (k^2 + 1)^2 = 0 \Leftrightarrow$$

$$\Leftrightarrow x = \pm\sqrt{3} \cdot (k^2 + 1) \cdot I$$

Consequently, for each value of k, except if $k = \pm I$, there are two different common points to the two circumferences:

$$(\sqrt{3} \cdot (k^2 + 1) \cdot I, 0) \quad , \quad (-\sqrt{3} \cdot (k^2 + 1) \cdot I, 0)$$

(i.e., if $k \neq 1$ then Theorem 1 is false).

Therefore, Theorem 1 is "almost always" false in the complex case.

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5 Conclusions

DGS and CAS are fruitful tools. But deducing results in plane geometry from DGS explorations or from blindly applying computer algebra techniques can lead to wrong conclusions. The key problem is that:

- we draw in $\mathbb{R}^2$,
- explore with a DGS in $\mathbb{R}^2$,
- intuitively think in $\mathbb{R}^2$,
- perform the effective computations in $\mathbb{C}^2$,
- think of generalizations in $\mathbb{C}^2$,

meanwhile the truthfulness of many theorems strongly depend on the base field considered.

As shown in the Introduction, it is very easy to find results that have solution or solutions if the base field considered is $\mathbb{C}^2$ but do not have a solution in $\mathbb{R}^2$.

What we have done here is different: we have presented a simple theorem (Theorem 1), which truthfulness discussion is very similar to that of MacLane 8_3 Theorem, but that is not a configuration theorem but a theorem which corresponding configuration is constructible.

That Theorem 1 is constructible makes an important difference. For instance, from the dynamic exploration point of view:

- in MacLane 8_3 Theorem the dynamic exploration cannot provide a certainty (only a guess) of the falseness of the theorem in $\mathbb{R}^2$ because we cannot be sure that we have tried all possible allowed configurations of the eight points,
- in Theorem 1 the dynamic exploration provides a certainty of its falseness in $\mathbb{R}^2$, as the theorem is constructive and only k can be changed.

Summarizing, the truthfulness of a geometric theorem may depend on the base field and has to be carefully analyzed even in simple looking constructive theorems.

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