International Journal of Computer Information Systems and Industrial Management Applications
ISSN 2150-7988 Volume 18(21s) 2026 pp. 1162-1176
https://doi.org/10.70917/ijcisim-2026-5385
Article

GEOMETRIC - ALPHA PROCESS AND A MAINTENANCE MODEL FOR DETERIORATING SYSTEM

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Abstract: The Geometric Process (GP) is a widely used stochastic model for analysing repairable systems with monotonic trends in successive inter-occurrence times. In this paper, we introduce the Geometric- Alpha Process (GAP), a new geometric-like process based on an alternative parametric formulation. Fundamental probabilistic properties of the GAP, including stochastic ordering and moment characteristics, are established, and a replacement model is formulated. The GAP is illustrated using the aircondit7 dataset, a benchmark reliability dataset containing operating times between failures of an aircraft air-conditioning system. For the analysed data, the GAP attains a higher maximized log-likelihood and a lower Akaike Information Criterion (AIC) than the classical GP, indicating a better fit.
Keywords: NA

1. Introduction

Repairable systems arise in many engineering applications, including manufacturing systems, aircraft, power plants, and communication networks. As these systems undergo successive failures and repairs, the distribution of inter-failure times may change because of deterioration, wear, or imperfect maintenance. Stochastic process models provide a natural framework for describing such changes and play an important role in reliability analysis, maintenance planning, and replacement decisions [8, 9, 18, 21].

The Geometric Process (GP), introduced by Lam [11,12], extends the renewal process by allowing successive inter-occurrence times to exhibit a monotone trend while preserving independence. Owing to its simple mathematical formulation and tractable probabilistic properties, the GP has been widely used in the analysis of repairable systems, maintenance optimisation, replacement policies, and warranty models [13,20]. Subsequent research has investigated its probabilistic properties, statistical inference, and applications, while several geometric-like processes have been proposed to broaden the class of models available for describing monotone stochastic behaviour [2,6,19,20].

Despite these developments, the classical GP is characterised by a fixed geometric scaling mechanism. In some applications, this formulation may not adequately represent the evolution of successive observations. This motivates the development of alternative scaling formulations within the geometric-process framework.

In this paper, we introduce the Geometric- Alpha Process (GAP), a new geometric-like process based on an alternative scaling formulation. We investigate its probabilistic properties and develop a replacement model under the proposed process.

The remainder of this paper is organized as follows. Section 2 presents the maintenance model assumptions and notation used throughout the study. Section 3 derives the long-run average cost function, while Section 4 establishes the optimal replacement policy under the proposed Geometric-Alpha Process. Section 5 provides numerical examples and sensitivity analysis, and Section 6 presents the empirical validation using the aircondit7 data set. Finally, Section 7 concludes the paper and outlines directions for future research.

Definition 1.1. A sequence of independent non-negative random variables X_n, ; $n = 1,2,3, \ldots$ is called a geometric process if the distribution function of X_n is $F\left(a^{n-1}x\right)$ for $n = 1,2,3, \ldots$, where a is a positive constant.

Definition 1.2. Let $\left\{X_j,\ j = 1,2,3, \ldots\right\}$ be a sequence of independent, non-negative random variables. The sequence constitutes a Geometric-Alpha Process (GAP) if there exists a positive parameter α such that the cumulative distribution function of X_j, denote as $F_j(x)$, is given by:

$$F_j(x) = F\left(\alpha^{j-1} j^{\alpha-1} x\right) \tag{1}$$

for $j = 1,2,3, \ldots$, where $F(x)$ is the baseline cumulative distribution function of X_1.

Definition 1.3. For two random variables Y and Z, Z is called stochastically less than Y (or Y is stochastically larger than Z) if for all real α,

$$P(Y > \alpha) \geq P(Z > \alpha). \tag{2}$$

This is written as $Z \leq_{st} Y$ or $Y \geq_{st} Z$.

Definition 1.4. A stochastic process $\{Y_n, n = 1,2,3, \ldots\}$ is said to be stochastically increasing (decreasing) if

$$Y_n \leq_{st} (\geq_{st}) Y_{n+1} \tag{3}$$

for all $n = 1,2,3, \ldots$.

Lemma 1.1. Given a Geometric-Alpha Process $\left\{X_j,\ j = 1,2,3, \ldots\right\}$,

(i) If $\alpha > 1$, then $\left\{X_j,\ j = 1,2,3, \ldots\right\}$ is stochastically decreasing.

(ii) If $0 < \alpha < 1$, then $\left\{X_j,\ j = 1,2,3, \ldots\right\}$ is stochastically increasing.

Proof. Let

$$A_j = \alpha^{j-1} j^{\alpha-1},\ j = 1,2,3, \ldots$$

(i) Suppose that $\alpha > 1$. Then

$$\frac{A_{j+1}}{A_j} = \frac{\alpha^j (j+1)^{\alpha-1}}{\alpha^{j-1} j^{\alpha-1}} = \alpha\left(\frac{j+1}{j}\right)^{\alpha-1} > 1,$$

since $\alpha > 1$ and $\left(\frac{j+1}{j}\right)^{\alpha-1} > 1$. Hence, $\left\{A_j\right\}$ is strictly increasing. Therefore, for every $x \geq 0$,

$$F(A_1 x) \leq F(A_2 x) \leq F(A_3 x) \leq \ldots \leq F(A_j x),$$

because F is a cumulative distribution function. Consequently,

$$P(X_1 > x) \geq P(X_2 > x) \geq P(X_3 > x) \geq \ldots \geq P(X_j > x),$$

which implies that

$$X_1 \geq_{st} X_2 \geq_{st} X_3 \geq_{st} \ldots \geq_{st} X_j.$$

Hence, $\{X_j,\ j = 1,2,3,\ldots\}$ is stochastically decreasing.

(ii)Suppose that $0 < \alpha < 1$. Then

$$\frac{A_{j+1}}{A_j} = \frac{\alpha^j (j+1)^{\alpha-1}}{\alpha^{j-1} j^{\alpha-1}} = \alpha \left(\frac{j+1}{j}\right)^{\alpha-1} < 1,$$

since $0 < \alpha < 1$ and $\left(\frac{j+1}{j}\right)^{\alpha-1} < 1$. Thus, $\{A_j\}$ is strictly decreasing. It follows that, for every $x \geq 0$,

$$F(A_1 x) \geq F(A_2 x) \geq F(A_3 x) \geq \ldots \geq F(A_j x),$$

and therefore

$$P(X_1 > x) \leq P(X_2 > x) \leq P(X_3 > x) \leq \ldots \leq P(X_j > x),$$

Hence,

$$X_1 \leq_{st} X_2 \leq_{st} X_3 \leq_{st} \ldots \leq_{st} X_j,$$

so $\{X_j,\ j = 1,2,3,\ldots\}$ is stochastically increasing. □

It is clear that if $\alpha = 1$, then the Geometric-Alpha Process is a renewal process.

Lemma 1.2. Let $E(X_1) = \lambda$ and $Var(X_1) = \sigma^2$. Then for $j = 1,2,3, \ldots,$

$$E(X_j) = \frac{\lambda}{\alpha^{j-1} j^{\alpha-1}} \text{ and } Var(X_j) = \frac{\sigma^2}{(\alpha^{j-1} j^{\alpha-1})^2} = \frac{\sigma^2}{\alpha^{2(j-1)} j^{2(\alpha-1)}}. \tag{4}$$

2. Model Descriptions

The maintenance framework for a deteriorating system is established upon the following postulates:

M1 A new system is commissioned at the start of the lifecycle. Upon failure, the system undergoes either repair or replacement with an equivalent unit.

M2 Let X_1 denote the initial operating time before the first failure with cumulative distribution function $F(x)$ and mean $E(X_1) = \lambda$. For $j = 1,2,3, \ldots,$ the subsequent operating times X_j follow the Geometric-Alpha Process with cumulative distribution function $F_j(x) = F\left(\alpha^{j-1} j^{\alpha-1} x\right)$, where $\alpha \geq 1$ is the deterioration parameter. For $\alpha > 1$, the successive operating times are stochastically decreasing, whereas for $\alpha = 1$, the process reduces to the classical renewal process.

M3 Let Y_1 denote the initial repair time after the first failure with cumulative distribution function $G(y)$ and mean $E(Y_1) = \mu$. For $j = 1,2,3, \ldots,$ the subsequent repair times Y_j have cumulative distribution function

$G_j(y) = G(\beta^{j-1} j^{\beta-1} y)$, where $0 < \beta \leq 1$ is the repair parameter. For $0 < \beta < 1$, the successive repair times are stochastically increasing, whereas for $\beta = 1$, the repair process reduces to the classical renewal process.

M4 Let Z be the replacement time with $E(Z) = \tau$.

M5 The process $\{X_n\}$, $\{Y_n\}$, and the variable Z are mutually independent.

M6 Let c denote the repair cost rate, r the reward rate, R the fixed replacement cost, and c_p the replacement cost rate associated with the replacement duration Z.

Table 1.Notations

Symbol	Description
X_j	Operating time between the $(j-1)$-th and j-th failures
X_1	Initial operating time before the first failure
$F(x)$	Cumulative distribution function (CDF) of the initial operating time X_1
$F_j(x)$	CDF of the operating time between the $(j-1)$-th and j-th failures under the GAP
λ	Mean initial operating time, $E(X_1)$
α	Deterioration parameter of the Geometric-Alpha Process
Y_j	Repair time after the j-th failure
Y_1	Initial repair time after the first failure
$G(y)$	Cumulative distribution function (CDF) of the initial repair time Y_1
$G_j(y)$	CDF of the repair time after the j-th failure under the GAP
μ	Mean initial repair time, $E(Y_1)$
β	Repair parameter of the Geometric-Alpha Process
Z	Replacement time of the system
τ	Mean replacement time, $E(Z)$
c	Repair cost incurred per unit repair time
r	Reward earned per unit operating time of the system
R	Replacement cost incurred per unit replacement time
c_p	Replacement cost rate (cost per unit replacement time)
$E(\cdot)$	Expectation operator
N	Replacement number (number of failures before replacement under the N-policy
$C(N)$	Expected long-run average cost per unit time under the N-policy
$B(N)$	Auxiliary function used to determine the optimal replacement policy

3. The Mean Cost in Long-run

The period between system initialization and the first replacement, or the duration separating two consecutive replacements, is defined as a cycle. These subsequent cycles establish a renewal process. Applying the renewal reward theorem, the long-run expected cost per unit time under an N-policy is the ratio of the total expected cost incurred within a cycle to the total expected duration of that cycle.

The mean cost $C(N)$ is derived as follows:

$$C(N) = \frac{\text{the mean cost accumulated during a cycle}}{\text{the mean cycle duration}}$$

$$= \frac{cE\left(\sum_{j=1}^{N-1} Y_j\right) + R + c_p E(Z) - rE\left(\sum_{j=1}^{N} X_j\right)}{E\left(\sum_{j=1}^{N} X_j\right) + E\left[\sum_{j=1}^{N-1} Y_j\right] + E(Z)}$$

$$= \frac{c\sum_{j=1}^{N-1} \frac{\mu}{\beta^{j-1} j^{\beta-1}} + R + c_p\tau - r\sum_{j=1}^{N} \frac{\lambda}{\alpha^{j-1} j^{\alpha-1}}}{\sum_{j=1}^{N} \frac{\lambda}{\alpha^{j-1} j^{\alpha-1}} + \sum_{j=1}^{N-1} \frac{\mu}{\beta^{j-1} j^{\beta-1}} + \tau}$$

$$C(N) = \frac{(c+r)\sum_{j=1}^{N-1} \frac{\mu}{\beta^{j-1} j^{\beta-1}} + R + \left(c_p + r\right)\tau}{\sum_{j=1}^{N} \frac{\lambda}{\alpha^{j-1} j^{\alpha-1}} + \sum_{j=1}^{N-1} \frac{\mu}{\beta^{j-1} j^{\beta-1}} + \tau} - r \tag{5}$$

4. The optimal Policy N^*

$$C(N+1) - C(N)$$

$$= \left[\frac{(c+r)\sum_{j=1}^{N} \frac{\mu}{\beta^{j-1} j^{\beta-1}} + R + \left(c_p + r\right)\tau}{\sum_{j=1}^{N+1} \frac{\lambda}{\alpha^{j-1} j^{\alpha-1}} + \sum_{j=1}^{N} \frac{\mu}{\beta^{j-1} j^{\beta-1}} + \tau} - \frac{(c+r)\sum_{j=1}^{N-1} \frac{\mu}{\beta^{j-1} j^{\beta-1}} + R + \left(c_p + r\right)\tau}{\sum_{j=1}^{N} \frac{\lambda}{\alpha^{j-1} j^{\alpha-1}} + \sum_{j=1}^{N-1} \frac{\mu}{\beta^{j-1} j^{\beta-1}} + \tau}\right]$$

$$= \frac{\begin{bmatrix} (c+r)\mu\lambda \frac{1}{\beta^{N-1} N^{\beta-1}} \sum_{j=1}^{N} \frac{1}{\alpha^{j-1} j^{\alpha-1}} - (c+r)\mu\lambda \frac{1}{\alpha^{N}(N+1)^{\alpha-1}} \sum_{j=1}^{N-1} \frac{1}{\beta^{j-1} j^{\beta-1}} \\ +(c+r)\mu\tau \frac{1}{\beta^{N-1} N^{\beta-1}} - R\lambda \frac{1}{\alpha^{N}(N+1)^{\alpha-1}} - R\mu \frac{1}{\beta^{N-1} N^{\beta-1}} - \left(c_p + r\right)\lambda\tau \frac{1}{\alpha^{N}(N+1)^{\alpha-1}} \\ -\left(c_p + r\right)\mu\tau \frac{1}{\beta^{N-1} N^{\beta-1}} \end{bmatrix}}{\left(\sum_{j=1}^{N+1} \frac{\lambda}{\alpha^{j-1} j^{\alpha-1}} + \sum_{j=1}^{N} \frac{\mu}{\beta^{j-1} j^{\beta-1}} + \tau\right)\left(\sum_{j=1}^{N} \frac{\lambda}{\alpha^{j-1} j^{\alpha-1}} + \sum_{j=1}^{N-1} \frac{\mu}{\beta^{j-1} j^{\beta-1}} + \tau\right)}$$

$$= \frac{\begin{bmatrix} (c+r)\mu\left[\frac{\lambda}{\beta^{N-1}\mathrm{N}^{\beta-1}}\sum_{j=1}^{N}\frac{1}{\alpha^{j-1}j^{\alpha-1}} - \frac{\lambda}{\alpha^{N}(N+1)^{\alpha-1}}\sum_{j=1}^{N-1}\frac{1}{\beta^{j-1}\mathrm{j}^{\beta-1}} + \frac{\tau}{\beta^{N-1}\mathrm{N}^{\beta-1}}\right] \\ -\left(R+\left(c_p+r\right)\tau\right)\left[\frac{\lambda}{\alpha^{N}(N+1)^{\alpha-1}} + \frac{\mu}{\beta^{N-1}\mathrm{N}^{\beta-1}}\right] \end{bmatrix}}{\mathrm{h}(N+1)\mathrm{h}(N)}$$

where

$$\mathrm{h}(N+1)\mathrm{h}(N)$$
$$= \left(\left(\sum_{\square=1}^{\square+1}\frac{\square}{\square^{\square-1}\square^{\square-1}} + \sum_{\square=1}^{\square}\frac{\square}{\square^{\square-1}\mathrm{j}^{\square-1}} + \square\right)\right.$$
$$\left.\times\left(\sum_{j=1}^{N}\frac{\lambda}{\alpha^{j-1}j^{\alpha-1}} + \sum_{j=1}^{N-1}\frac{\mu}{\beta^{j-1}\mathrm{j}^{\beta-1}} + \tau\right)\right)$$

$$= \frac{\begin{bmatrix} (c+r)\mu\left(\lambda\alpha^{N}(N+1)^{\alpha-1}\sum_{j=1}^{N}\frac{1}{\alpha^{j-1}j^{\alpha-1}} - \lambda\beta^{N-1}\mathrm{N}^{\beta-1}\sum_{j=1}^{N-1}\frac{1}{\beta^{j-1}\mathrm{j}^{\beta-1}} + \tau\alpha^{N}(N+1)^{\alpha-1}\right) \\ -\left(R+\left(c_p+r\right)\tau\right)\left[\lambda\beta^{N-1}\mathrm{N}^{\beta-1} + \mu\alpha^{N}(N+1)^{\alpha-1}\right] \end{bmatrix}}{\mathrm{h}(N+1)\mathrm{h}(N)\left(\left(\alpha^{N}(N+1)^{\alpha-1}\right)\left(\beta^{N-1}N^{\beta-1}\right)\right)}$$

let

$$B(N) = \frac{(c+r)\mu\left(\lambda\alpha^{N}(N+1)^{\alpha-1}\sum_{j=1}^{N}\frac{1}{\alpha^{j-1}j^{\alpha-1}} - \lambda\beta^{N-1}\mathrm{N}^{\beta-1}\sum_{j=1}^{N-1}\frac{1}{\beta^{j-1}\mathrm{j}^{\beta-1}} + \tau\alpha^{N}(N+1)^{\alpha-1}\right)}{\left(R+\left(c_p+r\right)\tau\right)\left[\lambda\beta^{N-1}\mathrm{N}^{\beta-1} + \mu\alpha^{N}(N+1)^{\alpha-1}\right]} \quad (6)$$

Lemma 4.1. For $C(N)$ and $B(N)$, the cost function $C(N)$ is increasing (decreasing) if and only if $B(N) > 1$ ($B(N) < 1$).

Proof. Since denominator is positive, the sign of $C(N+1) - C(N)$ depends solely on the numerator. Rearranging the inequality $C(N+1) - C(N) > 0$ according to the definition of $B(N)$ confirms that $C(N)$ is increasing if and only if $B(N) > 1$. The optimal policy N^* is the smallest integer N such that $B(N) \geq 1$.

Lemma 4.2. If $\alpha > 1$, then the sequence $\{\lambda_n\}$ is strictly decreasing.

Proof. For $n \geq 1$,

$$\frac{\lambda_{n+1}}{\lambda_n} = \frac{\lambda/(\alpha^{n}(n+1)^{\alpha-1})}{\lambda/(\alpha^{n-1}n^{\alpha-1})} = \frac{1}{\alpha}\left(\frac{n}{n+1}\right)^{\alpha-1}.$$

Since $\alpha > 1$,

$\frac{1}{\alpha}$ and $\left(\frac{n}{n+1}\right)^{\alpha-1} < 1$,

it follows that

$$\frac{\lambda_{n+1}}{\lambda_n} < 1.$$

Hence,

$$\lambda_{n+1} < \lambda_n$$

for all $n \geq 1$. Therefore, the sequence $\{\lambda_n\}$ is strictly decreasing. □

Similarly, for $0 < \beta < 1$, the sequence $\{\mu_n\}$ is strictly increasing.

Lemma 4.3. The auxiliary function $B(N)$ is non-decreasing in N.

From equation (6),

$$B(N+1) - B(N)$$

$$= \left[\frac{(c+r)\mu \begin{bmatrix} \lambda\alpha^{N+1}(N+2)^{\alpha-1}\sum_{j=1}^{N+1}\frac{1}{\alpha^{j-1}j^{\alpha-1}} - \lambda\beta^{N}(N+1)^{\beta-1}\sum_{j=1}^{N}\frac{1}{\beta^{j-1}j^{\beta-1}} \\ +\tau\alpha^{N+1}(N+2)^{\alpha-1} \end{bmatrix}}{\left(R+\left(c_p+r\right)\tau\right)[\lambda\beta^{N}(N+1)^{\beta-1}+\mu\alpha^{N+1}(N+2)^{\alpha-1}]} - \frac{(c+r)\mu \begin{bmatrix} \lambda\alpha^{N}(N+1)^{\alpha-1}\sum_{j=1}^{N}\frac{1}{\alpha^{j-1}j^{\alpha-1}} - \lambda\beta^{N-1}N^{\beta-1}\sum_{j=1}^{N-1}\frac{1}{\beta^{j-1}j^{\beta-1}} \\ +\tau\alpha^{N}(N+1)^{\alpha-1} \end{bmatrix}}{\left(R+\left(c_p+r\right)\tau\right)[\lambda\beta^{N-1}N^{\beta-1}+\mu\alpha^{N}(N+1)^{\alpha-1}]} \right]$$

$$= \frac{(c+r)\mu}{\left(R+\left(c_p+r\right)\tau\right)} \frac{\begin{bmatrix} \lambda^2\sum_{j=1}^{N}\frac{1}{\alpha^{j-1}j^{\alpha-1}} \\ \left(\alpha^{N+1}(N+2)^{\alpha-1}\beta^{N-1}N^{\beta-1} - \beta^{N}(N+1)^{\beta-1}\alpha^{N}(N+1)^{\alpha-1}\right) \\ +\left(\frac{\lambda^2}{\alpha^{N}(N+1)^{\alpha-1}}\right) \\ \left(\alpha^{N+1}(N+2)^{\alpha-1}\beta^{N-1}N^{\beta-1} - \beta^{N}(N+1)^{\beta-1}\alpha^{N}(N+1)^{\alpha-1}\right) \\ +\frac{\mu\lambda}{\beta^{N-1}N^{\beta-1}} \\ \left(\alpha^{N+1}(N+2)^{\alpha-1}\beta^{N-1}N^{\beta-1} - \beta^{N}(N+1)^{\beta-1}\alpha^{N}(N+1)^{\alpha-1}\right) \\ +\mu\lambda\sum_{j=1}^{N-1}\frac{1}{\beta^{j-1}j^{\beta-1}} \\ \left(\alpha^{N+1}(N+2)^{\alpha-1}\beta^{N-1}N^{\beta-1} - \beta^{N}(N+1)^{\beta-1}\alpha^{N}(N+1)^{\alpha-1}\right) \\ +\lambda\tau\left(\alpha^{N+1}(N+2)^{\alpha-1}\beta^{N-1}N^{\beta-1} - \beta^{N}(N+1)^{\beta-1}\alpha^{N}(N+1)^{\alpha-1}\right) \end{bmatrix}}{\left(\lambda\beta^{N-1}N^{\beta-1}+\mu\alpha^{N}(N+1)^{\alpha-1}\right)\left(\lambda\beta^{N}(N+1)^{\beta-1}+\mu\alpha^{N+1}(N+2)^{\alpha-1}\right)}$$

$$
= \frac{(c+r)\mu \begin{bmatrix} \left(\lambda\beta^{N-1}N^{\beta-1} + \mu\alpha^{N}(N+1)^{\alpha-1}\right) \\ \left(\begin{array}{l} \lambda\alpha^{N+1}(N+2)^{\alpha-1}\sum_{j=1}^{N+1}\frac{1}{\alpha^{j-1}j^{\alpha-1}} \\ -\lambda\beta^{N}(N+1)^{\beta-1}\sum_{j=1}^{N}\frac{1}{\beta^{j-1}j^{\beta-1}} + \tau\alpha^{N+1}(N+2)^{\alpha-1} \end{array} \right) \\ -\left(\lambda\beta^{N}(N+1)^{\beta-1} + \mu\alpha^{N+1}(N+2)^{\alpha-1}\right) \\ \left(\begin{array}{l} \lambda\alpha^{N}(N+1)^{\alpha-1}\sum_{j=1}^{N}\frac{1}{\alpha^{j-1}j^{\alpha-1}} \\ -\lambda\beta^{N-1}N^{\beta-1}\sum_{j=1}^{N-1}\frac{1}{\beta^{j-1}j^{\beta-1}} + \tau\alpha^{N}(N+1)^{\alpha-1} \end{array} \right) \end{bmatrix}}{\left(R + (c_p + r)\tau\right)\left(\lambda\beta^{N-1}N^{\beta-1} + \mu\alpha^{N}(N+1)^{\alpha-1}\right)\left(\lambda\beta^{N}(N+1)^{\beta-1} + \mu\alpha^{N+1}(N+2)^{\alpha-1}\right)}
$$

$$
= \frac{\begin{bmatrix} (c+r)\mu\lambda\left(\alpha^{N+1}(N+2)^{\alpha-1}\beta^{N-1}N^{\beta-1} - \beta^{N}(N+1)^{\beta-1}\alpha^{N}(N+1)^{\alpha-1}\right) \\ \left(\sum_{j=1}^{N}\frac{\lambda}{\alpha^{j-1}j^{\alpha-1}} + \frac{\lambda}{\alpha^{N}(N+1)^{\alpha-1}} + \frac{\mu}{\beta^{N-1}N^{\beta-1}} + \sum_{j=1}^{N-1}\frac{\mu}{\beta^{j-1}j^{\beta-1}} + \tau\right) \end{bmatrix}}{\left(R + (c_p + r)\tau\right)\left(\lambda\beta^{N-1}N^{\beta-1} + \mu\alpha^{N}(N+1)^{\alpha-1}\right)\left(\lambda\beta^{N}(N+1)^{\beta-1} + \mu\alpha^{N+1}(N+2)^{\alpha-1}\right)}
$$

This implies that $B(N)$ is non-decreasing in N, because λ_n is decreasing in n and μ_n is increasing in n.

Theorem 4.1 For the GAP, the optimal replacement policy N^* is given by

$$
N^* = min\{N: B(N) \geq 1\}. \tag{7}
$$

This policy N^* is unique if and only if $B(N^*) > 1$.

Proof. By lemma 4.1 and 4.3, $C(N)$ decreases for $N < N^*$ and increases for $N \geq N^*$, confirming N^* as the minimum of $C(N)$, with $B(N^*) > 1$ ensuring uniqueness.

5. Numerical Examples and Sensitivity Analysis

To examine the performance of the proposed maintenance model, a numerical study is carried out using a fixed set of parameter values as the reference setting. Unless otherwise specified, the parameters are fixed at $\alpha = 1.05, \beta = 0.95,\ R = 5000,\ c_p = 10, \lambda = 40, \mu = 15,\ r = 50, \tau = 10, c = 10$. In the sensitivity analysis, one parameter is varied while all other parameters are kept unchanged. This allows the effect of each parameter on the optimal replacement policy to be examined independently. The parameters α, R, and λ are selected because they represent the deterioration behaviour, replacement cost, and operating performance of the system, respectively.

5.1 Effect of the deterioration parameter $\boldsymbol{\alpha}$

The effect of the deterioration parameter α on the optimal replacement policy is first investigated. Three values, namely $\alpha = 1.00,\ 1.05,$ and $1.10,$ are considered to represent the renewal case, moderate deterioration, and faster deterioration, while all remaining parameters are fixed at their reference values.

Table 2. Effect of the deterioration parameter α on the optimal replacement policy

$\alpha = 1.00$			$\alpha = 1.05$			$\alpha = 1.10$		
N	C(N)	B(N)	N	C(N)	B(N)	N	C(N)	B(N)
1	62.0000	0.146104	1	62.0000	0.155137	1	62.0000	0.164234
2	11.9048	0.165653	2	13.8522	0.191018	2	15.7040	0.216231
3	-3.6353	0.191509	3	-0.9412	0.240218	3	1.6296	0.287621
4	-11.0137	0.224577	4	-7.8103	0.303458	4	-4.7526	0.378020
5	-15.2110	0.265395	5	-11.5775	0.380984	5	-8.1164	0.486229
6	-17.8423	0.314402	6	-13.8119	0.472780	6	-9.9887	0.610471
7	-19.5874	0.371989	7	-15.1761	0.578610	7	-11.0167	0.748514
8	-20.7820	0.438518	8	-15.9984	0.698027	8	-11.5228	0.897801
9	-21.6105	0.514321	9	-16.4599	0.830391	**9**	**-11.6857**	**1.055594**
10	-22.1828	0.599707	10	-16.6690	0.974884	10	-11.6133	1.219117
11	-22.5683	0.694960	**11**	**-16.6946**	**1.130535**	11	-11.3746	1.385684
12	-22.8129	0.800337	12	-16.5827	1.296246	12	-11.0162	1.552804
13	-22.9483	0.916068	13	-16.3653	1.470819	13	-10.5706	1.718261
14	**-22.9971**	**1.042355**	14	-16.0655	1.652986	14	-10.0616	1.880162
15	-22.9757	1.179373	15	-15.7006	1.841442	15	-9.5069	2.036955
16	-22.8966	1.327264	16	-15.2835	2.034873	16	-8.9200	2.187431
17	-22.7690	1.486143	17	-14.8245	2.231982	17	-8.3114	2.330701
18	-22.6005	1.656091	18	-14.3317	2.431516	18	-7.6895	2.466165
19	-22.3967	1.837160	19	-13.8119	2.632280	19	-7.0607	2.593476
20	-22.1623	2.029368	20	-13.2704	2.833162	20	-6.4306	2.712498

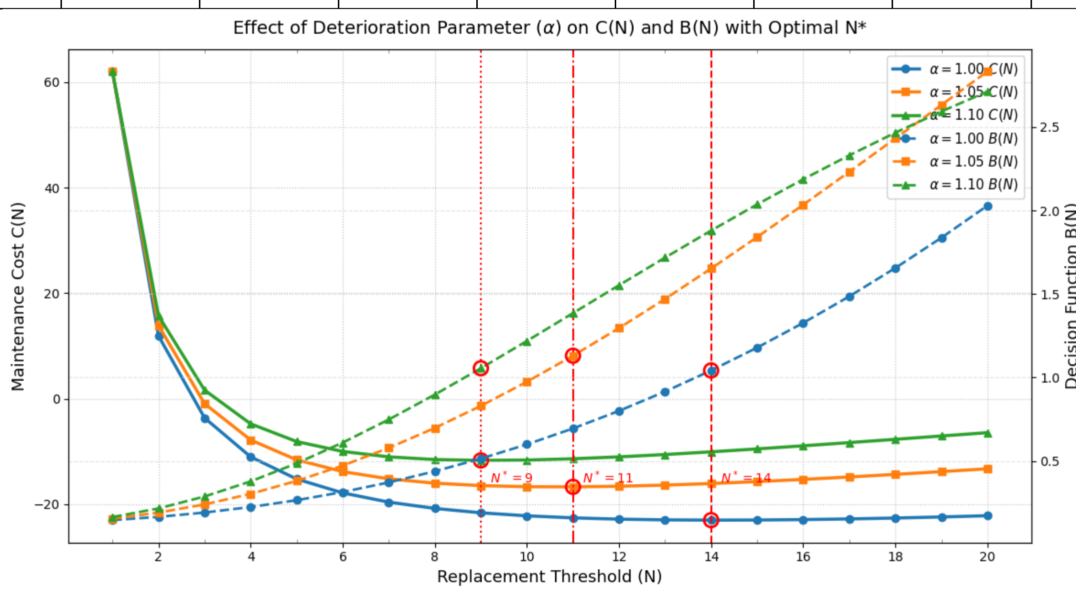

Figure 1. Variation of C(N) and B(N) for different values of the deterioration parameter α

As shown in Table 2, the optimal replacement number decreases from $N^* = 14$ to $N^* = 11$ and further to $N^* = 9$ as α increases from 1.00 to 1.10. This indicates that a higher deterioration rate reduces the economically desirable operating life of the system, making earlier replacement preferable.

Figure 1 illustrates the corresponding changes in the long-run average cost $C(N)$ and the auxiliary function $B(N)$ for different values of α.

The graphical results are consistent with those presented in Table 2. As the deterioration rate increases, the minimum cost is attained at smaller values of N, while the behaviour of B(N) identifies the same optimal replacement policy.

5.2 Effect of the replacement cost R

The second study investigates the effect of the replacement cost R on the optimal replacement policy. The replacement cost is varied from 4000 to 6000, while all other model parameters are kept at their reference values. This analysis illustrates how changes in replacement expenditure influence the economically optimal replacement decision.

Table 3. Effect of the replacement cost R on the optimal replacement policy

R=4000			**R=5000**			**R=6000**		
N	C(N)	B(N)	N	C(N)	B(N)	N	C(N)	B(N)
1	42.0000	0.188862	1	62.0000	0.155137	1	82.0000	0.131631
2	4.0288	0.232544	2	13.8522	0.191018	2	23.6756	0.162076
3	-7.4992	0.292439	3	-0.9412	0.240218	3	5.6168	0.203821
4	-12.7538	0.369427	4	-7.8103	0.303458	4	-2.8668	0.257479
5	-15.5553	0.463806	5	-11.5775	0.380984	5	-7.5998	0.323259
6	-17.1451	0.575559	6	-13.8119	0.472780	6	-10.4787	0.401147
7	-18.0472	0.704395	7	-15.1761	0.578610	7	-12.3050	0.490942
8	-18.5210	0.849772	8	-15.9984	0.698027	8	-13.4757	0.592265
9	**-18.7096**	**1.010910**	9	-16.4599	0.830391	9	-14.2102	0.704574
10	-18.6986	1.186815	10	-16.6690	0.974884	10	-14.6394	0.827174
11	-18.5423	1.376304	**11**	**-16.6946**	**1.130535**	11	-14.8469	0.959242
12	-18.2773	1.578039	12	-16.5827	1.296246	**12**	**-14.8880**	**1.099845**
13	-17.9289	1.790562	13	-16.3653	1.470819	13	-14.8017	1.247967
14	-17.5154	2.012331	14	-16.0655	1.652986	14	-14.6156	1.402533
15	-17.0507	2.241756	15	-15.7006	1.841442	15	-14.3505	1.562436
16	-16.5451	2.477237	16	-15.2835	2.034873	16	-14.0219	1.726559
17	-16.0069	2.717196	17	-14.8245	2.231982	17	-13.6421	1.893803
18	-15.4427	2.960106	18	-14.3317	2.431516	18	-13.2208	2.063104
19	-14.8580	3.204515	19	-13.8119	2.632280	19	-12.7658	2.233450
20	-14.2573	3.449067	20	-13.2704	2.833162	20	-12.2835	2.403895

As shown in Table 3, the optimal replacement number increases from $N^* = 9$ to $N^* = 11$ and then to $N^* = 12$ as the replacement cost increases. The observed change is gradual, indicating that although a higher replacement cost encourages slightly longer system operation before replacement, its influence on the optimal policy is relatively moderate within the range considered.

Figure 2 illustrates the variation of the long-run average cost C(N) and the auxiliary function B(N) for different values of the replacement cost R.

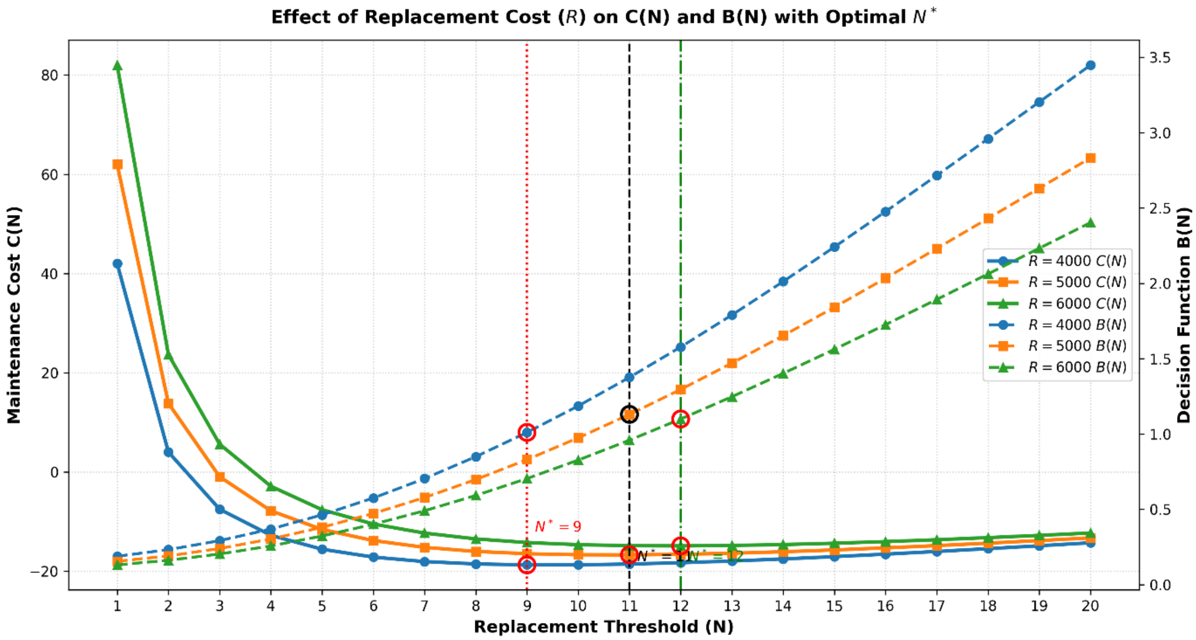

Figure 2. Variation of C(N) and B(N) for different values of the replacement cost R

The graphical results support the observations obtained from Table 3. As the replacement cost increases, the minimum cost occurs at slightly larger replacement numbers. However, the overall shift is relatively small, suggesting that moderate changes in replacement cost do not substantially alter the optimal replacement policy.

5.3 Effect of the mean operating time λ

The final numerical study examines the influence of the mean operating time λ on the optimal replacement policy. The values of λ are varied while all remaining model parameters are fixed at their reference values. Since λ represents the expected operating time before failure, this analysis illustrates how changes in system operating performance affect the replacement decision.

Table 4. Effect of the mean operating time λ on the optimal replacement policy

$\lambda = 35$			$\lambda = 40$			$\lambda = 45$		
N	C(N)	B(N)	N	C(N)	B(N)	N	C(N)	B(N)
1	74.4444	0.153230	1	62.0000	0.155137	1	51.8182	0.156733
2	20.5005	0.187324	2	13.8522	0.191018	2	8.3497	0.194135
3	3.9764	0.233836	3	-0.9412	0.240218	3	-5.0375	0.245644
4	-3.7089	0.293305	4	-7.8103	0.303458	4	-11.2441	0.312154

5	-7.9444	0.365812	5	-11.5775	0.380984	5	-14.6328	0.394073
6	-10.4791	0.451197	6	-13.8119	0.472780	6	-16.6262	0.491535
7	-12.0507	0.549095	7	-15.1761	0.578610	7	-17.8259	0.604441
8	-13.0240	0.658959	8	-15.9984	0.698027	8	-18.5299	0.732461
9	-13.6004	0.780076	9	-16.4599	0.830391	9	-18.9029	0.875047
10	-13.9001	0.911590	10	-16.6690	0.974884	**10**	**-19.0434**	**1.031447**
11	**-13.9996**	**1.052527**	**11**	**-16.6946**	**1.130535**	11	-19.0142	1.200719
12	-13.9498	1.201821	12	-16.5827	1.296246	12	-18.8571	1.381761
13	-13.7863	1.358338	13	-16.3653	1.470819	13	-18.6013	1.573336
14	-13.5346	1.520912	14	-16.0655	1.652986	14	-18.2679	1.774107
15	-13.2136	1.688361	15	-15.7006	1.841442	15	-17.8726	1.982666
16	-12.8376	1.859520	16	-15.2835	2.034873	16	-17.4273	2.197569
17	-12.4179	2.033258	17	-14.8245	2.231982	17	-16.9414	2.417369
18	-11.9634	2.208500	18	-14.3317	2.431516	18	-16.4224	2.640643
19	-11.4813	2.384237	19	-13.8119	2.632280	19	-15.8765	2.866020
20	-10.9774	2.559541	20	-13.2704	2.833162	20	-15.3089	3.092202

Table 4 shows that the optimal replacement number remains unchanged at $N^* = 11$ for $\lambda = 35$ and $\lambda = 40$, although the corresponding values of C(N) and B(N) differ. When the mean operating time increases to $\lambda = 45$, the optimal replacement number changes slightly to $N^* = 10$. These results indicate that moderate changes in the mean operating time mainly influence the maintenance cost, while the optimal replacement policy remains relatively stable over a certain range of λ.

Figure 3 illustrates the variation of the long-run average cost C(N) and the auxiliary function B(N) for different values of the mean operating time λ.

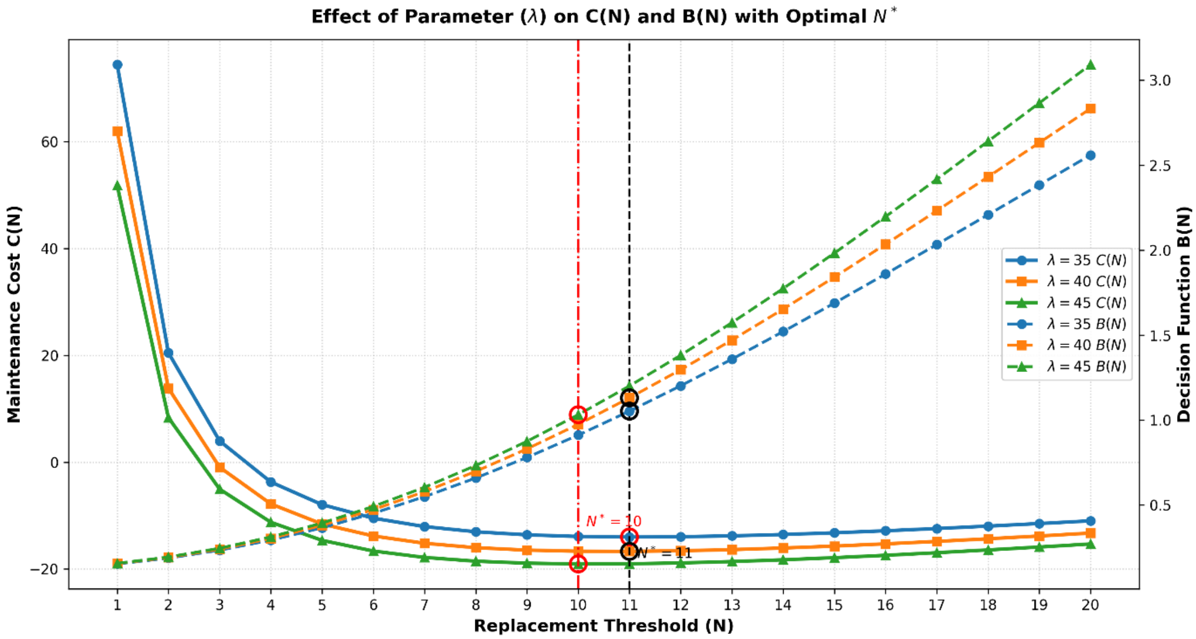

Figure 3. Variation of C(N) and B(N) for different values of the mean operating time λ

The graphical results agree with the observations in Table 4. Although the cost curves vary with changes in λ, the location of the minimum cost remains almost unchanged for $\lambda = 35$ and $\lambda = 40$. A noticeable shift in the optimal replacement point is observed only when λ increases to 45, indicating that the replacement policy is relatively insensitive to small changes in the mean operating time within the range considered.

6. Empirical Validation

For empirical validation, we analyse the aircondit7 dataset from the boot package in R. The data consist of successive operating times between failures of an aircraft air-conditioning system and are widely used for evaluating stochastic process models in reliability.

The parameters of the proposed GAP were estimated using the maximum likelihood method under a Weibull baseline distribution. The optimization was performed using the L-BFGS-B algorithm subject to the admissible parameter constraints and converged successfully. The resulting maximum likelihood estimates, together with their standard errors and 95% confidence intervals, are reported in Table 5. These confidence intervals quantify the uncertainty associated with the estimated parameters.

The fitted GAP achieved a maximized log-likelihood of −78.9563, corresponding to an Akaike Information Criterion (AIC) of 163.9126.

Table 5. Maximum likelihood estimates of the GAP parameters with standard errors and 95% confidence intervals.

Parameter	Estimate	Standard Error	95% CI Lower	95% CI Upper
Shape	7.2244	1.2375	4.7989	9.6499
Scale	6.3079	0.3972	5.5293	7.0865
α	0.8721	0.0035	0.8652	0.8789

For comparison, the standard Geometric Process (GP) was fitted to the same data using the identical Weibull baseline distribution. The log-likelihood, AIC, ΔAIC, and Akaike weights for both models are presented in Table 6.

Table 6. Comparison of the GAP and the standard GP.

Model	Log-likelihood	AIC	ΔAIC	Akaike Weight
GAP	**-78.9563**	**163.9126**	**0.0000**	**0.8826**
GP	-80.9732	167.9463	4.0337	0.1174

The GAP achieved a higher maximized log-likelihood and a lower AIC than the standard GP. The AIC difference of 4.0337 favours the GAP over the standard GP. The corresponding Akaike weights indicate that the GAP receives approximately 88.3% of the model support, whereas the GP receives 11.7%. These findings indicate that, for the aircondit7 data set, the proposed GAP provides a better fit than the standard GP under the same Weibull baseline distribution.

7. Conclusion

This study introduced the Geometric-Alpha Process (GAP) as a stage-dependent extension of the classical Geometric Process for modelling the operating and repair behaviour of deteriorating systems.

For the aircondit7 data set, the GAP was fitted using a Weibull baseline distribution and compared with the classical Geometric Process under the same baseline distribution. The GAP resulted in a better fit for the data set considered. The conditions for stochastic increase and stochastic decrease of the GAP were established, and expressions for the expectation and variance of its successive random variables were derived.

An N-failure replacement policy was developed under the proposed process. The long-run average cost per unit time was obtained in closed form, and the optimal replacement number was determined by minimizing this cost. For the parameter settings considered in the numerical analysis, changes in the deterioration parameters, replacement cost, and mean operating time resulted in changes in the optimal replacement number.

Further research may consider extensions of the proposed model to imperfect repair, preventive replacement, warranty models, and multi-component repairable systems.

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