International Journal of Computer Information Systems and Industrial Management Applications
ISSN 2150-7988 Volume 18(23s) 2026 pp. 1165-1169
https://doi.org/10.70917/ijcisim-2026-5690
Article

OUTER CONNECTED GEODETIC NUMBER OF SOME GRAPHS

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Abstract: For a connected graph G of order at least two, an outer connected geodetic set S of G is a geodetic set such that S = V(G) or the subgraph induced by V(G) − S is connected. The minimum cardinality of an outer connected geodetic set of G and is denoted by $g_{o}c(G)$. We determine in this paper, the outer connected geodetic number for some special classes of graphs.
Keywords: Geodetic set, Outer connected geodetic set, outer connected geodetic number

Mathematics Subject Classification: 05C12

1. Introduction

By a graph $G = (V, E)$, we mean a finite simple undirected connected graph. The order and size of G are denoted by m and n, respectively. For base theoretic terminology, we refer to Harary [1,8]. For any two vertices u and v in a connected graph G, the distance $d(x, y)$ is the length of a shortest $u - v$ in G. A $u - v$ path of length $d(x, y)$ is called $u - v$ geodesic. A vertex v of G is said to lie on a $u - v$ geodesic P if v is a vertex of P including the vertices u and v. For any vertex u of G, the eccentricity of u is defined as $e(u) = \{d(u, v); v \in V(G)\}$. The radius $r(G)$ and diameter $diam(G)$ of G are defined as $r(G) = \{e(v); v \in V(G)\}$, respectively. The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v of G is called an extreme vertex of G if the subgraph induced by its neighbours is complete.

The closed interval $I[x, y]$ consists of all vertices lying on some $u - v$ geodesic of G while for $S \subseteq V(G), I[S] = \cup_{x,y \in S} I[x, y]$. A set $S \subseteq V(G)$ is said to be a geodetic set of G if $I[S] = V(G)$. The minimum cardinality of a geodetic set of G is the geodetic number $g(G)$ of G. The geodetic number of a graph and its variants have been studied by several authors in [6,7,9,10].

The following Theorems will be used in the sequel.

Theorem 1.1 [7]: Each extreme vertex of a connected graph G belongs to every geodetic set of G.

2. Outer Connected Geodetic Number

Definition: 2.1

A set $S \subseteq V(G)$ is said to be an outer connected geodetic set of G if S is a geodetic set of G and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected geodetic set of G is the outer connected geodetic number of G and is denoted by $g_{oc}(G)$.

Example 2.2

For the graph G, given in Figure 2.1, it is clear that $S = \{v_1, v_5\}$ is the unique minimum geodetic set of G and so $g(G) = 2$. But the subgraph induced by $V - S$ is not connected. Therefore, S is not an outer connected geodetic set of G.

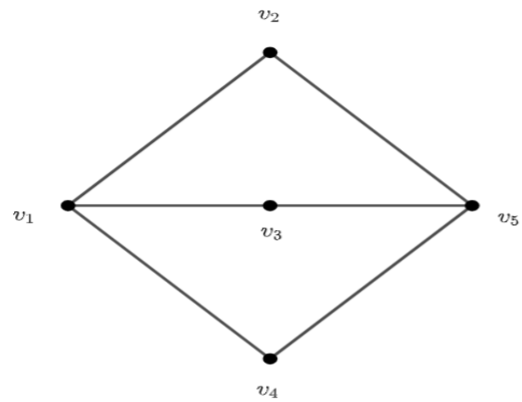

Figure 2.1

Now it is clear that $S \cup \{v_2, v_3\}$ is a minimum outer connected geodetic set of G with cardinality 4. Thus $g_{oc}(G) = 4$.

Hence the geodetic number and the outer connected geodetic number of G are different.

Theorem 2.3

Each extreme vertex of a connected graph G belongs to every outer connected geodetic set of G.

Proof

Since we know by definition every outer connected geodetic set of G is also a geodetic set of G. Therefore by Theorem 1.1, each extreme vertex of G belongs to every outer connected geodetic set of G.

Corollary 2.4

For any complete graph $K_n (n \geq 2), g_{oc}(K_n) = n$.

Proof

All the vertices of K_n are of degree $n - 1$. Therefore, they are extreme vertices and so by Theorem 2.3, $g_{oc}(K_n) \geq n$. Since the set of all vertices of K_n form an outer connected geodetic set of K_n and so $g_{oc}(K_n) \leq n$. Thus, $g_{oc}(K_n) = n$.

Theorem 2.5

For any star graph $K_{1,n-1} (n \geq 2), g_{oc}(K_{1,n-1}) = \{n - 1 \; if \; n \geq 3 \;\; n \; if \; n = 2$.

Proof

For $n = 2, K_{1,n-1} \cong K_{1,1} \cong K_2$ and so by Theorem 2.5, $g_{oc}(K_{1,1}) = 2 = n.$ Now, assume $n \geq 3.$ Let $V(K_{1,n-1}) = \{x, x_1, x_2, \dots, x_{n-1}\}$, where x is of degree $n-1$ and each x_i for $1 \leq i \leq n-1$ one of degree 1. Therefore each x_i are extreme vertices. So by Theorem 2.3, $S = \{x_1, x_2 \dots, x_{n-1}\}$ must be in every outer connected geodetic set of $K_{1,n-1}$. Therefore $g_{oc}(K_{1,n-1}) \geq |S| = n-1.$ Since x lies on every $x_i - x_j$ geodesic path of S and the $V(K_{1,n-1}) - S$ contains only one vertex that $< V(K_{1,n-1}) - S >$ is connected, hence S itself an outer connected geodetic set of $K_{1,n-1}$ and so $g_{oc}(K_{1,n-1}) \leq |S| = n-1.$ Then $g_{oc}(K_{1,n-1}) = n-1.$

Therefore, $g_{oc}(K_{1,n-1}) = \{n-1 \ if \ n \geq 3 \ \ n \ if \ n = 2$.

Theorem 2.6

For the bistar graph $B_{m,n}(m, n \geq 1), g_{oc}(B_{m,n}) = m + n.$

Proof

Let $G = B_{m,n}$ be a bistar graph of order $m + n + 2.$ Let $V(G) = \{x, y, x_i, y_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{xy, xx_i, yy_j; 1 \leq i \leq m, 1 \leq j \leq n\}.$ Let S be a minimum outer connected geodetic set of G. So by Theorem 2.3, $\{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n\} \subseteq S.$ Since x and y lies on every geodetic between vertices from S and x, y are adjacent, it follows that $S = \{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n\}.$ Thus, we conclude that $g_{oc}(G) = m + n.$

Theorem 2.7

For the path graph $P_n(n \geq 2), g_{oc}(P_n) = n.$

Proof

Let P_n be a path graph of order $n \geq 2$ with $V(P_n) = \{x_1, x_2, \dots, x_n\}.$ If $n \geq 2,$ then $P_n \cong K_2$ and so by corollary 2.4, $g_{oc}(P_2) = 2.$ Now assume $n \geq 3.$ Let $S = \{x_1, x_n\}.$ Clearly every vertex of P_n must lie on the geodesic path between x_1 and x_n and so that S is a geodesic set of P_n. Also, $< V(P_n) - S >= P_{n-2}$, which is a path. Therefore the subgraph induced by $V(P_n) - S$ is connected and so that S is an outer connected geodetic set of P_n and so $g_{oc}(P_n) \leq |S| = 2.$ But, for every outer connected geodetic set needs minimum two vertices $g_{oc}(P_n) \geq 2.$ Hence $g_{oc}(P_n) = 2.$

Theorem 2.8

For the cycle graph $C_n(n \geq 4), g_{oc}(C_n) = \{\frac{n}{2} + 1 \ if \ n \ is \ even \ \lfloor\frac{n}{2}\rfloor + 2 \ if \ n \ is \ odd$.

Proof

Let C_n be a cycle graph of order $n \geq 4.$ Here, $V(C_n) = \{x_1, x_2, \dots, x_n\}.$ We prove this theorem by considering two cases.

Case (i) Suppose n is even.

Let $n = 2k,$ where $k \geq 2.$ Let $C_{2k}: x_1 x_2 \dots x_{2k} x_1$ be a cycle of order $2k.$ Consider $S = \{x_1, x_{k+1}\}.$ Clearly every vertex in $V(C_{2k})$ lies on some geodesic between vertices from S and so that S is a geodetic set of C_{2k}. But the subgraph induced by $V(C_{2k}) - S$ is disconnected. Now, take $S_1 = S \cup \{x_2, x_3, \dots, x_k\}$. It is clear that S is an outer connected geodetic set of C_{2k} and so $g_{oc}(C_{2k}) \leq k + 1$. Now, it can be easily verified that there is no outer connected geodetic set of C_{2k} exist with fever than $k + 1$ vertices. Thus $g_{oc}(C_{2k}) = k + 1 = \frac{n}{2} + 1$.

Case (ii) n is odd.

Let $n = 2k + 1$, where $k \geq 2$. Let $C_{2k+1}: x_1 x_2 \dots x_{2k} x_{2k+1} x_1$ be a cycle graph of order $2k + 1$. Consider $S = \{x_1, x_{k+1}, x_{k+2}\}$. Clearly every vertex of $V(C_{2k+1})$ lies on some geodesic path between vertices from S. So that S is a geodetic set of C_{2k+1}. But the subgraph induced by $V(C_{2k+1}) - S$ is not connected. Therefore, that S is not an outer connected geodetic set of C_{2k+1}. Now take $S_1 = S \cup \{x_2, x_3, \dots, x_k\}$. It can be easily referred that S_1 is an outer connected geodetic set of C_{2k+1} and so $g_{oc}(C_{2k+1}) \leq |S| = k + 2$. Now, it is clear that there does not exist an outer connected geodetic set of cardinality less than $k + 2$ vertices. Thus, $g_{oc}(C_{2k+1}) = k + 2 = \lfloor \frac{n}{2} \rfloor + 2$.

Hence, $g_{oc}(C_n) = \{\frac{n}{2} + 1 \; if \; n \; is \; even \; \lfloor \frac{n}{2} \rfloor + 2 \; if \; n \; is \; odd$.

Theorem 2.9

If G is a connected graph with a cut vertex v, then every outer connected geodetic set of G contains at least one vertex from each component $G - v$.

Proof

Let G be a connected graph with a cut vertex v. Let $G_1, G_2, \dots, G_k (k \geq 2)$ be the components of $G - v$. Let S be an outer connected geodetic set of G. On the contrary, we assume that S contain no vertex from a component say $G_i (1 \leq i \leq k)$. Let u be any vertex on G_i. Then by Theorem 2.3, u is not an extreme vertex of G. Since S is an outer connected geodetic set of G, there exist vertices x, y in S such that u lies on an $x - y$ geodesic path $P: x = u_0, u_1, u_2, \dots, u, \dots, u_l = y$ such that $u \neq x, y$. Since v is a cut vertex of G, that subpath $x - y$ of P both contains the vertex v. It shows that P is not a path, which is a contradiction. Hence, every outer connected geodetic set of G contains at least one vertex from each component of $G - v$.

Corollary 2.10

Let G be a connected graph with cut vertices and let S be an outer connected geodetic set of G. Then every branch of G contains an element of S.

Theorem 2.11

No cut vertex of a connected graph G belongs to a minimum outer connected geodetic set of G.

Proof

Let G be a connected graph with a cut vertex v and let S be any minimum outer connected geodetic set of G. Then by Theorem 2.9, every component of $G - v$ contains an element of S. Suppose $v \in S$. Let X and Y be any two disjoint components of $G - v$. Let $x \in X$ and $y \in Y$. Then v is an internal vertex of an $x - y$ geodesic path. Now, consider $S' = S - \{v\}$. Then it can be easily verified that every internal vertex lies on an $x - v$ geodesic path also lies on an $x - y$ geodesic path. It follows that S' is not an outer connected geodetic set of G with cardinality less than S. Therefore, that S is not a minimum outer connected geodetic set of G, which is contradiction. Hence, $v \notin S$.

Corollary 2.12

For any tree T with k −end vertices $g_{oc}(T) = k$.

Proof

Let T be a tree with k end vertices. Therefore, there are k extreme vertices and the remaining are cut vertices so by Theorem 2.3, $g_{oc}(T) \geq k$. Also, by Theorem 2.11, $g_{oc}(T) \leq k$. Hence, $g_{oc}(T) = k$.

Theorem 2.13

For any connected graph G of order $n \geq 2$, $g_{oc}(G) = n$ if and only if $G = K_n$.

Proof

Let G be a connected graph of order $n \geq 2$. First assume, $g_{oc}(G) = n$. To prove $G = K_n$. Suppose G is not complete. Then there exist any two vertices x, y such that they are distinct and non-adjacent in G. Since G is connected, there is an $u - v$ geodesic path from x to y as P and $d(u, v) \geq 2$ in G. Let z be a vertex in P such that $x \neq u, v$. This is possible because $d(u, v) \geq 2$. Then it is clear that $S = V(G) - \{z\}$ is an outer connected geodetic set of G and so $g_{oc}(G) \leq |S| = n - 1$, which is a contradiction. Hence $G = K_n$. Conversely, assume $G = K_n$. Then, by Corollary 2.4, $g_{oc}(G) = n$.

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