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Article

ON THE OUTER CONNECTED GEO CHROMATIC NUMBER OF SOME GRAPHS

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Abstract: For a connected graph G of order at least two, an outer connected geo chromatic set S of G is a geo chromatic set such that S = V(G) or the subgraph induced by V(G) − S is connected. The minimum cardinality of an outer connected geo chromatic set of G is the outer connected geo chromatic number of G and is denoted by $\chi_{ogc}(G)$. In this paper, we determine the outer connected geo chromatic number for some special graph and its bounds.

Keywords: Geo chromatic set, outer connected geo chromatic set, outer connected geo chromatic number

Mathematics Subject Classification: 05C12

1. Introduction

By a graph $G = (V, E)$, we mean a finite simple undirected connected graph. The order and size of G are denoted by m and n, respectively. For basic theoretic terminology, we refer to Harary [1,8]. For any two vertices u and v in a connected graph G, the distance $d(x, y)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(x, y)$ is called $u - v$ geodesic. A vertex v of G is said to lie on a $u - v$ geodesic P if v is a vertex of P including the vertices u and v. For any vertex u of G, the eccentricity of u is defined as $e(u) = \{d(u, v); v \in V(G)\}$. The radius $r(G)$ and diameter $diam(G)$ of G are defined as $r(G) = min\{e(v); v \in V(G)\}$ and $diam(G) = max\{e(v); v \in V(G)\}$ respectively. The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v of G is called an extreme vertex of G if the subgraph induced by its neighbours is complete.

The closed interval $I[x, y]$ consider of all vertices lying on some $u - v$ geodesic of G, while for some $S \subseteq V(G), I[S] = \cup_{x,y \in S} I[x, y]$. A set S of vertices of G is a geodetic set of G if $I[S] = V(G)$ and the minimum cardinality of a geodetic set of G is the geodetic number of G and is denoted by $g(G)$. A set $S \subseteq V(G)$ is called the chromatic set of G if S contains all the vertices of distinct colours in G. The chromatic number $\chi(G)$ of G is the minimum cardinality among all the chromatic sets in G. A set $S \subseteq V(G)$ is called a geo chromatic set of G if S is both a geodetic set and a chromatic set of G. The minimum cardinality of a geo chromatic set of G is the geo chromatic number of G and is denoted by $\chi_g(G)$ [ ]. In this paper, we define the outer connected geo chromatic number of a graph and studied some properties.

Throughout this paper, we consider G as a connected graph with at least two vertices.

Theorem 1.1 [5]

Each extreme vertex of a connected graph G belongs to every geo chromatic set of G.

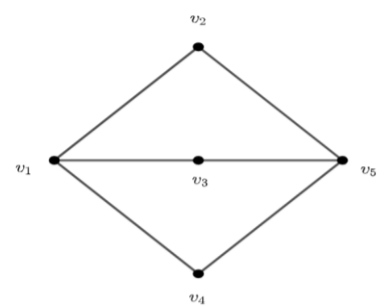
2. Outer Connected Geo Chromatic Number of Graphs

Definition 2.1

A set S of geo chromatic set of G is an outer connected geo chromatic set of G if either $S = V(G)$ or the subgraph induced by $V(G) - S$ is connected. The minimum cardinality of an outer connected geo chromatic set of G is the outer connected geo chromatic number of G and is denoted by $\chi_{g_{oc}}(G)$.

Example 2.2

For the graph G given in Figure 2.1, it is clear that $S = \{v_1, v_5\}$ is the unique minimum geodetic set of G and $S_1 = \{v_1, v_2\}$ is a minimum chromatic set of G and so $g(G) = \chi(G) = 2.$ But S is not a chromatic set of G as well as S_1 is not a geodetic set of G. Therefore, that S and S_1 itself is not a geo chromatic set of G. Now it is clear that $S_2 = \{v_1, v_2, v_5\}$ is a minimum geo chromatic set of G and so $\chi_g(G) = |S_2| = 3.$ But $S_2 \neq V(G)$ and the subgraph induced by $V(G) - S_2$ is not connected. Therefore that S_3 is not an outer connected geo chromatic set of G.



G $Figure$ 2.1

It can be easily verified that $S_3 = \{v_1, v_2, v_3, v_5\}$ is a minimum outer connected geo chromatic set of G and so $\chi_{g_{oc}}(G) = |S_3| = 4.$

Thus the outer connected geo chromatic number is different from geodetic number chromatic number and geo chromatic number of G.

Theorem 2.3

There can be more than one minimum outer connected geo chromatic set of G.

For example, given in Figure 2.1, $S_3 = \{v_1, v_2, v_3, v_5\}, S_4 = \{v_1, v_3, v_4, v_5\}, S_5 = \{v_1, v_2, v_4, v_5\}, S_6 = \{v_1, v_2, v_3, v_4\}, S_7 = \{v_2, v_3, v_4, v_5\}$ are the minimum order connected geo chromatic sets of G. Therefore, minimum outer connected geo chromatic set need not be unique.

Theorem 2.4

Let G be a connected graph of order $n \geq 2.$ Then $2 \leq \chi_g(G) \leq \chi_{g_{oc}}(G) \leq n.$

Proof

Let G be a connected graph of order $n \geq 2.$ Any geo chromatic set of G needs minimum two vertices, we have $\chi_g(G) \geq 2.$ Also, every outer connected geo chromatic set of G is a geo chromatic set, it follows that $\chi_g(G) \leq \chi_{g_{oc}}(G).$ Moreover, the set of all vertices of G form an outer connected geo chromatic set of G, we have $\chi_{g_{oc}}(G) \leq n.$

Hence, $2 \leq \chi_g(G) \leq \chi_{g_{oc}}(G) \leq n.$

Remark 2.5

All the inequalities in Theorem 2.4 are can be strict. For the graph G, given in Figure 2.1, $\chi_g(G) = 3, \chi_{g_{oc}}(G) = 4$ and $n = 5.$ Thus, $2 < \chi_g(G) < \chi_{g_{oc}}(G) < n.$

Corollary 2.6

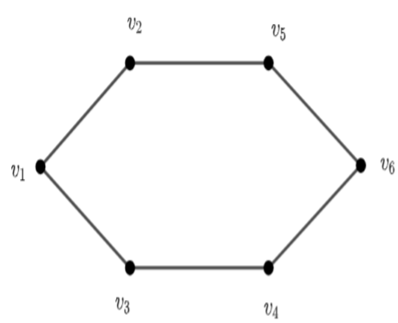
Let G be a connected graph. If $\chi_{g_{oc}}(G) = 2,$ then $\chi_g(G) = 2.$

Proof

Let G be a connected graph. Assume $\chi_{g_{oc}}(G) = 2.$ Then ny Theorem 2.4, $2 \leq \chi_g(G) \leq 2.$ This implies that $\chi_g(G) = 2.$

Remark 2.7

The converse of corollary 2.6 need not be true. For consider the graph G given in Figure 2.2.



$G\ Figure$ 2.2

Here $S = \{v_1, v_6\}$ is a minimum geo chromatic set of G and so $\chi_g(G) = 2.$ But $S_1 = S \cup \{v_2, v_5\}$ is a minimum outer connected geo chromatic set of G and so $\chi_{g_{oc}}(G) = 4.$

Theorem 2.8

Each extreme vertex of a connected graph G belongs to every outer connected geo chromatic set of G.

Proof

Since every outer connected geo chromatic is also a geo chromatic set of G, the theorem follows from Theorem 1.1.

Corollary 2.9

For the complete graph $K_n(n \geq 2), \chi_{g_{oc}}(K_n) = n.$

Proof

Since the complete graph K_n contains only the extreme vertices and K_n contains a vertices, $\chi_{g_{oc}}(K_n) \geq n.$ By Theorem 2.4, we conclude that $\chi_{g_{oc}}(K_n) = n.$

Theorem 2.10

Let G be any connected graph and if the set of all extreme vertices S of G is an outer connected geo chromatic set of G, then S is the unique minimum outer connected geo chromatic set of G.

Proof

Let G be a connected graph and let G be the set of all extreme vertices of G, which is an outer connected geo chromatic set of G. Then $\chi_{g_{oc}}(G) \leq |S|.$ But by Theorem 2.4, $\chi_{g_{oc}}(G) \geq |S|.$ Thus $\chi_{g_{oc}}(G) = |S|$ it follows that S is a minimum outer connected geo chromatic set of G. Moreover, by Theorem 2.4, every minimum outer connected geo chromatic set contains S and so that S is the unique minimum outer connected geo chromatic set of G.

Observation 2.11

Let G be a connected graph. Then the following are true.

(i) A minimum outer connected geodetic set S of G, which is also a chromatic set of G, then S is a minimum outer connected geo chromatic set of G.

(ii) A minimum chromatic set S of G, which is also an outer connected geodetic set of G, then S is a minimum outer connected geo chromatic set of G.

Theorem 2.12

Let G be a connected with cut vertices and let S be an outer connected geo chromatic set of G. If v is a cut vertex of G, then every component of $G - v$ contains an element of S.

Proof

Let v be a cut vertex of G and let S be an outer connected geo chromatic set of G. To prove every component of $G - v$ contains an element of S. Suppose there exists a component, say G_1 of $G - v$ such that G_1 contains no vertices of S. Then by Theorem 2.4, S contains all the extreme vertices of G. This shows that G_1 contains no extreme vertices of $G.$ Let $u \in V(G_1).$ (This is possible because G is connected). Since S is an outer connected geo chromatic set of G, there exist $x, y \in S$ with v is an internal vertex of some $x - y$ geodetic path $P: x = u_0, u_1, u_2, \dots, v, \dots, u_l = y$ in G. Since v is a cut vertex of G, the $x - v$ subpath of P both contains $u.$ It follows that P is not a path, which is a contradiction. Hence every component of $G - v$ contains an element of S.

Corollary 2.13

Let G be a connected graph with cut vertices and let S be an outer connected geo chromatic set of G. Then every branch of G contains an element of S.

Proof

Let G be a connected graph and let S be an outer connected geo chromatic set of G. Let v be a cut vertex of G. Since every branch of G at v is a component of $G - v$ with the vertex v together all the edges joining at x to $V(G_1)$, by Theorem 2.12, we conclude that every branch of G contains an element of S.

Theorem 2.14

Let G be any connected graph and let $S \subseteq V(G)$ with S is clique in G. Then S is an outer connected geo chromatic set of G if and only if $S = V(G)$.

Proof

Let G be any connected graph and $S \subseteq V(G)$ be a complete subgraph, that is a clique in G. First assume S is an outer connected geo chromatic set of G. To prove $S = V(G)$. Suppose on the contrary, that $S \neq V(G)$. This means that $S \subset V(G)$. Let $v \in V(G) - S$. Since the subgraph induced by S is complete, $d(x, y) = 1$ for every $x, y \in S$. It follows that v does not lie on any geodetic path between any pair of vertices from S. This shows that S is not an outer connected geo chromatic set of G, which is a contradiction. Thus, $S = V(G)$. Conversely, assume that $S = V(G)$. We know the set of all vertices form an outer connected geo chromatic set of G. Hence, that S is an outer connected geo chromatic set of G.

Corollary 2.15

Let G be a non-complete connected graph. Then every outer connected geo chromatic set of G contains at least two non-adjacent vertices.

Proof

Let G be a non-complete connected graph. Since G is non-complete, that G has at least two non-adjacent vertices. Let S be an outer connected geo chromatic set of G. To prove that S contains at least two non-adjacent vertices. If $S = V(G)$, then by hypothesis that S contains at least two non-adjacent vertices. Now suppose $S \neq V(G)$. To prove S contains at least two non-adjacent vertices. Suppose on the contrary, that S does not contain any non-adjacent vertices of G. It follows that the subgraph of G induced by S is complete. Thus that S is a clique in G. Therefore, by Theorem 2.14, that $S = V(G)$, which is a contradiction. Hence, every outer connected geo chromatic set of G has at least two non-adjacent vertices.

Theorem 2.16

For any positive integer $2 \leq a \leq n$, there exists a connected graph G of order n and $\chi_{g_{oc}}(G) = a$.

Proof

For $a = n$, we take G as the complete graph of n vertices. Then by Corollary 2.9 $\chi_{g_{oc}}(G) = n$.

For $2 \leq a < n$, we consider the connected graph G as shown in Figure 2.3.

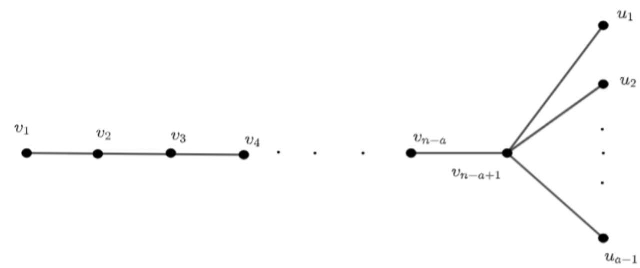


Figure 2.3: G

Here $S = \{v_1, u_1, u_2, \ldots, u_{a-1}\}$ be the set of all extreme vertices of G. By Theorem 2.8, $\chi_{g_{oc}}(G) \geq a.$ Now it is clear that S itself is an outer connected geo chromatic set of G and so $\chi_{g_{oc}}(G) \leq a.$ Hence, $\chi_{g_{oc}}(G) = a.$

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