

QUOTIENT SQUARE SUM SP E- CORDIAL LABELING

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Abstract: Graph labeling is a well-established area of graph theory in which numbers are assigned to the vertices or edges of a graph subject to certain conditions. Cordial labeling, introduced by Cahit, requires the induced vertex and edge label classes to differ by at most one, and has since given rise to numerous variants, including SP (sum-product) based E-cordial labelings. In this paper, we introduce a new variant called Quotient Square Sum SP E-Cordial Labeling, in which each edge of a graph is assigned one of two labels, and the induced label of every vertex is derived from an arithmetic function that combines the sum and product of the labels of its incident edges, taking the quotient of the sum of their squares to their sum, and reducing the result modulo two. A graph is said to be Quotient Square Sum SP E-Cordial if the resulting vertex and edge label distributions differ by at most one in each class. In this paper, we establish that several standard graph families the path, cycle, comb graph, crown graph, triangular snake, and the tadpole (dragon) graph admit this labeling. Explicit labeling schemes are constructed for each family, and the induced vertex and edge partitions are computed and shown to satisfy the required cordiality conditions in all cases. We further extend these results to establish that the helm graph, closed helm graph, semi-total point graph of a path, semi-total point graph of a cycle, subdivision of a star graph, and the count tree graph also admit this labeling, bringing the total number of graph families treated in this paper to twelve.

Keywords: Graph labeling; Cordial labeling; E-Cordial labeling; SP labeling; Quotient Square Sum labeling; Path graph; Cycle graph; Comb graph; Crown graph; Triangular snake; Tadpole graph; Dragon graph

1. Introduction

Graph labeling has emerged as one of the most actively studied areas of graph theory since Rosa introduced the notion of β -valuations (later called graceful labelings) in 1967. Since then, numerous labeling schemes — graceful, harmonious, magic, and cordial labelings among others — have been developed, each imposing a distinct arithmetic condition relating vertex and edge labels. Cordial labeling, introduced by Cahit in 1987 as a weaker form of graceful and harmonious labeling, assigns binary labels $\{0, 1\}$ to the vertices of a graph such that the induced edge labels (obtained via absolute difference) satisfy a near-balance condition: the number of vertices (respectively edges) labeled 0 and 1 differ by at most one. This relatively simple balancing requirement has made cordial labeling a rich framework, admitting numerous variants — including E-cordial labeling, product cordial labeling, total cordial labeling, and SP (sum-product) based cordial labelings — each extending the basic idea to different combinations of vertex and edge arithmetic.

In this paper, we introduce and study a new SP-type variant, the **Quotient Square Sum SP E-Cordial Labeling**. Unlike classical cordial labeling, where vertex labels are assigned directly and edge labels are induced, our



scheme reverses the direction: edges are labeled from the set $\{2\}$, and the vertex labels are induced through an arithmetic function combining the sum S and product P of the incident edge labels. Specifically, the induced vertex label is obtained by taking the floor of the quotient of the sum of squares $S^2 + P^2$ to the sum $S + P$, reduced modulo 2. This construction produces a nontrivial, non-monotonic relationship between the local edge environment of a vertex and its induced binary label, making the resulting labeling patterns of independent combinatorial interest.

We investigate the existence of Quotient Square Sum SP E-Cordial labelings for several fundamental graph families that frequently serve as building blocks in labeling theory: the path P_n , the cycle C_n , the comb graph $P_n \odot K_1$ (a path with a pendant vertex at every node), the crown graph $C_n \odot K_1$ (a cycle with a pendant vertex at every node), the triangular snake nC_3 (a chain of triangles obtained by identifying consecutive edges of a path with new apex vertices), and the tadpole (dragon) graph $T_{n,n}$ (a cycle joined to a pendant path by a bridge edge). For each family, we construct an explicit edge labeling, derive the induced vertex partition, and verify that the resulting distributions satisfy the required cordiality bounds $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(1) - e_f(2)| \leq 1$.

The remainder of this paper is organized as follows. Section 2 recalls the basic definitions and notation used throughout, including the formal definition of Quotient Square Sum SP E-Cordial labeling. Section 3 presents the main results, establishing that the path, cycle, comb, crown, triangular snake, and tadpole graphs each admit this labeling, with complete proofs and induced label distributions. Section 4 concludes with a discussion of open problems and directions for further extending this labeling scheme to other graph families.

Definition

Let G be a simple graph. Let f be a function $f: E \rightarrow \{1, 2\}$ and an induced function $f^*: V \rightarrow \{0, 1\}$. We associate two integers $S = \sum_{f(uv) \in E(G)} f(uv)$ and $P = \prod_{f(uv) \in E(G)} f(uv)$. For each vertex u assign the label as follows $f(u) = \lfloor \frac{S^2 + P^2}{S + P} \rfloor \pmod{2}$ then f is called quotient square sum SP E-cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(1) - e_f(2)| \leq 1$ where $v_f(0)$ and $v_f(1)$ is the number of vertices labeled with 0 and labeled by 1 and $e_f(1)$ and $e_f(2)$ is the number of edges labeled with 1 and labeled by 2 respectively. A graph which admits quotient square sum SP E-cordial labeling is called quotient square sum SP E-cordial graph.

Theorem 1.

The Path graph P_n is a quotient square sum SP E-cordial graph

Proof:

Let $V(P_n) = \{u_1, u_2, \dots, u_n\}$ be the vertices and $E(P_n) = \{u_1u_2, u_2u_3, u_3u_4, \dots, u_{n-1}u_n\}$ be the edges. Then $|V(P_n)| = n$ and $|E(P_n)| = n - 1$

Case: 1 When n is odd

To define the labelling function $f: E(G) \rightarrow \{1, 2\}$ is defined as follows

$$f(u_i u_{i+1}) = \begin{cases} 1 & \text{if } 1 \leq i \leq \frac{n-1}{2} \\ 2 & \text{if } \frac{n-1}{2} < i \leq n-1 \end{cases}$$

Case: 2 When n is even

$$f(u_i u_{i+1}) = \begin{cases} 1 & \text{if } 1 \leq i \leq \frac{n}{2} \\ 2 & \text{if } \frac{n}{2} < i \leq n-1 \end{cases}$$

Find $S = \sum_{uv \in E(G)} f(uv)$ and $P = \prod_{uv \in E(G)} f(uv)$. for each vertex in $V(P_n)$

Define $f^*: V(G) \rightarrow \{0,1\}$ defined by $f(u) = \lfloor \frac{S^2+P^2}{S+P} \rfloor \pmod{2}$

The numerical distribution obtained for the path graph is summarized in Table 1.

	$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
n is odd $n \geq 3$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$
n is even $n \geq 4$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n-2}{2}$

Table 1. Distribution of induced vertex labels and edge labels for path graph (P_n)

The above assignment gives, $|e_f(1) - e_f(2)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$

Hence path graph P_n admits Quotient square sum SP E-cordial labelling.

Theorem 2.

The Cycle graph C_n is a Quotient square sum SP E-cordial graph if n is odd.

Proof:

Let $V(C_n) = \{u_1, u_2, \dots, u_n\}$ be the vertices and $E(C_n) = \{u_1u_2, u_2u_3, u_3u_4, \dots, u_{n-1}u_n, u_nu_1\}$ be the edges.

Then $|V(C_n)| = n$ and $|E(C_n)| = n$

when n is odd

To define the labelling function $f: E(G) \rightarrow \{1,2\}$ is defined as follows

$$f(u_iu_{i+1}) = \begin{cases} 1 & \text{if } 1 \leq i \leq \frac{n+1}{2} \\ 2 & \text{if } \frac{n+1}{2} < i \leq n \end{cases}$$

and assign $f(u_nu_1) = 2$.

Find $S = \sum_{uv \in E(G)} f(uv)$ and $P = \prod_{uv \in E(G)} f(uv)$. for each vertex in $V(C_n)$

Define $f^*: V(G) \rightarrow \{0,1\}$ defined by $f(u) = \lfloor \frac{S^2+P^2}{S+P} \rfloor \pmod{2}$

The resulting label counts for the cycle graph are listed in Table 2.

	$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
n is odd	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$

Table 2. Induced label distribution for cycle graph (C_n)

Hence the induced vertex and edge labels satisfy, $|e_f(1) - e_f(2)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$

Therefore, cycle graph C_n is a Quotient square sum SP E-cordial graph.

Theorem 3.

The Comb graph $P_n \odot K_1$ is a Quotient square sum SP E-cordial graph

Proof:

Let $V(P_n \odot K_1) = \{u_i, u'_i | 1 \leq i \leq n\}$ be the vertices and $E(P_n \odot K_1) = \{u_i u_{i+1}, u u'_i, 1 \leq i \leq n\}$ be the edges. Then $|V(P_n \odot K_1)| = 2n$ and $|E(P_n \odot K_1)| = 2n - 1$

To define the labelling function $f: E(G) \rightarrow \{1, 2\}$ is defined as follows

$$f(u_i u_{i+1}) = 2, 1 \leq i \leq n - 1$$

$$f(u_i u'_i) = 1, 1 \leq i \leq n$$

$$\text{Find } S = \sum_{uv \in E(G)} f(uv) \text{ and } P = \prod_{uv \in E(G)} f(uv). \text{ for each vertex in } V(P_n \odot K_1)$$

$$\text{Define } f^*: V(G) \rightarrow \{0, 1\} \text{ defined by } f(u) = \left\lfloor \frac{S^2 + P^2}{S + P} \right\rfloor \pmod{2}$$

The obtained distribution for the comb graph is presented in Table 3.

$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
n	n	n	$n - 1$

Table 3. Label arrangement for comb graph $(P_n \odot K_1)$

Thus, the constructed labelling fulfils $|v_f(0) - v_f(1)| \leq 1$ together with

$$|e_f(1) - e_f(2)| \leq 1.$$

Consequently, comb graph $P_n \odot K_1$ admits the required labelling.

Theorem 4.

The Crown graph $C_n \odot K_1$ is a Quotient square sum SP E-cordial graph

Proof:

Let $V(C_n \odot K_1) = \{u_1, u_2, u_3, \dots, u_n, u'_1, u'_2, u'_3, \dots, u'_n\}$ be the vertices and $E(C_n \odot K_1) = \{u_1 u_2, u_2 u_3, u_3 u_4, \dots, u_n u_1, u_1 u'_1, u_2 u'_2, u_3 u'_3, \dots, u_n u'_n\}$ be the edges.

$$\text{Then } |V(C_n \odot K_1)| = 2n \text{ and } |E(C_n \odot K_1)| = 2n$$

To define the labelling function $f: E(G) \rightarrow \{1, 2\}$ is defined as follows

$$\text{For } 1 \leq i \leq n - 1$$

$$f(u_i u_{i+1}) = 2, 1 \leq i \leq n - 1, f(u_n u_1) = 2,$$

$$f(u_i u'_i) = 1, 1 \leq i \leq n$$

$$\text{Find } S = \sum_{uv \in E(G)} f(uv) \text{ and } P = \prod_{uv \in E(G)} f(uv). \text{ for each vertex in } V(C_n \odot K_1)$$

$$\text{Define } f^*: V(G) \rightarrow \{0, 1\} \text{ defined by } f(u) = \left\lfloor \frac{S^2 + P^2}{S + P} \right\rfloor \pmod{2}$$

The balanced label counts for the crown graph are shown in Table 4.

$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
n	n	n	n

Table 4. Distribution of labels for crown graph $(C_n \odot K_1)$

Therefore, the required vertex and edge distributions are obtained:

$$|v_f(0) - v_f(1)| \leq 1 \text{ and } |e_f(1) - e_f(2)| \leq 1.$$

Hence Crown graph $C_n \odot K_1$ possesses Quotient Square sum SP E-cordial labelling.

Theorem 5.

The Triangular snake nC_3 is a Quotient square sum SP E-cordial graph

Proof:

Let the path P_n having the vertices $u_1, u_2, u_3, \dots, u_n$ and the edges $e_1, e_2, e_3, \dots, e_{n-1}$.

To construct Triangular snake nC_3 from path P_n join u_i and u_{i+1} to a new edge v_i by edges $u_i v_i$ and $u_{i+1} v_i$, for $i=1, 2, 3, \dots, n-1$.

Then $|V(nC_3)| = 2n - 1$ and $|E(nC_3)| = 3n - 3$

To define the labelling function $f: E(G) \rightarrow \{1, 2\}$

$$f(u_i u_{i+1}) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

$$f(u_i v_i) = 1$$

$$f(u_{i+1} v_i) = 2$$

Find $S = \sum_{uv \in E(G)} f(uv)$ and $P = \prod_{uv \in E(G)} f(uv)$. for each vertex in $V(nC_3)$

Define $f^*: V(G) \rightarrow \{0, 1\}$ defined by $f(u) = \left\lfloor \frac{S^2 + P^2}{S + P} \right\rfloor \pmod{2}$

The computed label distribution for the triangular snake graph is given in Table 5.

	$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
$n \text{ is odd}$	n	$n - 1$	$\frac{3(n-1)}{2}$	$\frac{3(n-1)}{2}$
$n \text{ is even}$	n	$n - 1$	$\frac{3n-2}{2}$	$\frac{3n-4}{2}$

Table 5. Vertex-edge label distribution for triangular snake graph nC_3

The obtained labels satisfy the inequalities $|v_f(0) - v_f(1)| \leq 1$ and

$$|e_f(1) - e_f(2)| \leq 1.$$

Thus, Triangular snake nC_3 is proved to be a Quotient square sum SP E-cordial graph.

Theorem 6.

The Helm graph H_n is a Quotient square sum SP E-cordial graph if n is even

Proof:

Let $V(H_n) = \{u, u_1, u_2, u_3, \dots, u_n, u'_1, u'_2, u'_3, \dots, u'_n\}$ be the vertices and $E(H_n) = \{uu_1, uu_2, uu_3, \dots, uu_n, u_1u_2, u_2u_3, u_3u_4, \dots, u_nu_1, u_1u'_1, u_2u'_2, \dots, u_nu'_n\}$ be the edges. Here u is apex vertex and $u_1, u_2, u_3, \dots, u_n$ be the vertices of cycle C_n and $u'_1, u'_2, u'_3, \dots, u'_n$ are pendent vertices adjacent to $u_1, u_2, u_3, \dots, u_n$ respectively.

Then $|V(H_n)| = 2n + 1$ and $|E(H_n)| = 3n$

To define the labelling function $f: E(G) \rightarrow \{1, 2\}$ is defined as follows

$$f(uu_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

$$f(u_iu_{i+1}) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

$$f(u_iu'_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

Find $S = \sum_{uv \in E(G)} f(uv)$ and $P = \prod_{uv \in E(G)} f(uv)$. for each vertex in $V(H_n)$

Define $f^*: V(G) \rightarrow \{0, 1\}$ defined by $f(u) = \left\lfloor \frac{S^2 + P^2}{S + P} \right\rfloor \pmod{2}$

The corresponding balanced labels for the helm graph are displayed in Table 6.

	$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
If $f^*(u) = 1$	n	$n+1$	$\frac{3n}{2}$	$\frac{3n}{2}$
If $f^*(u) = 0$	$n+1$	n	$\frac{3n}{2}$	$\frac{3n}{2}$

Table 6. Label distribution for helm graph (H_n)

Accordingly, the balance requirements become $|v_f(0) - v_f(1)| \leq 1$ and

Consequently, helm graph satisfies the required cordial conditions.

Theorem 7.

The Semi total point graph of cycle $T_2(C_n)$ is a Quotient square sum SP E-cordial graph if n is even

Proof:

Let the path C_n having the vertices $u_1, u_2, u_3, \dots, u_n$ and the edges $e_1, e_2, e_3, \dots, e_{n-1}$.

To construct Semi total point graph of cycle $T_2(C_n)$ from path P_n join u_i and u_{i+1} to a new edge e_i by edges $u_i e_i$ and $u_{i+1} e_i$, for $i=1, 2, 3, \dots, n-1$.

Then $|V(T_2(C_n))| = 2n$ and $|E(T_2(C_n))| = 3n$

To define the labelling function $f: E(G) \rightarrow \{1, 2\}$

$$f(u_i u_{i+1}) = \{1 \text{ if } i \text{ is odd } 2 \text{ if } i \text{ is even}\}$$

$$f(u_i u'_i) = 1$$

$$f(u_{i+1} u'_i) = 2$$

Find $S = \sum_{uv \in E(G)} f(uv)$ and $P = \prod_{uv \in E(G)} f(uv)$. for each vertex in $V(T_2(C_n))$

Define $f^*: V(G) \rightarrow \{0,1\}$ defined by $f(u) = \left\lfloor \frac{S^2 + P^2}{S + P} \right\rfloor \pmod{2}$

The resulting label configuration for the semi-total point graph is shown in Table 7.

$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
n	n	$\frac{3n}{2}$	$\frac{3n}{2}$

Table 7. Label distribution for semi-total point graph $T_2(C_n)$

Thus, both inequalities are satisfied: $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(1) - e_f(2)| \leq 1$.

Therefore, semi-total point graph of cycle $T_2(C_n)$ is a quotient square sum SP E-cordial graph.

Theorem 8.

The Closed Helm graph CH_n is a Quotient square sum SP E-cordial graph

Proof:

Let u be apex vertex $u_1, u_2, u_3, \dots, u_n$ be the vertices of inner cycle and $v_1, v_2, v_3, \dots, v_n$ be the vertices of outer cycle of closed helm CH_n .

$$|V(CH_n)| = 2n + 1 \text{ and } |E(CH_n)| = 4n$$

We define $f: E(CH_n) \rightarrow \{1,2\}$ as follows

$$f(u_i u) = 1, 1 \leq i \leq n.$$

$$f(u_i u_{i+1}) = 2, 1 \leq i \leq n - 1, f(u_n u_1) = 2$$

$$f(u_i v_i) = 1, 1 \leq i \leq n.$$

$$f(v_i v_{i+1}) = 2, 1 \leq i \leq n - 1, f(v_n v_1) = 2$$

Find $S = \sum_{uv \in E(G)} f(uv)$ and $P = \prod_{uv \in E(G)} f(uv)$. for each vertex in $V(CH_n)$

Define $f^*: V(G) \rightarrow \{0,1\}$ defined by $f(u) = \left\lfloor \frac{S^2 + P^2}{S + P} \right\rfloor \pmod{2}$

The obtained distribution for the closed helm graph is presented in Table 8 .

	$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
$n \text{ is even}$	n	$n + 1$	$2n$	$2n$

n is odd	$n + 1$	n	$2n$	$2n$
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Table 8. Label distribution for closed helm graph CH_n

In each case labeling defined as above satisfies $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(1) - e_f(2)| \leq 1$. Hence Closed Helm CH_n admits quotient square sum SP E-cordial labeling.

Closed Helm CH_n is a quotient square sum SP E-cordial graph.

Theorem 9.

The Subdivision of Star graph $S(S_n)$ is a Quotient Square Sum SP E-Cordial graph, for every even n .

Proof:

Let $V[S(S_n)] = \{u, u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$, where u is the apex, $u_1, \dots, u_n$ are the subdivided star-arm vertices, and $v_1, \dots, v_n$ are the pendant vertices attached to $u_1, \dots, u_n$ respectively.

$$E[S(S_n)] = \{uu_1, uu_2, \dots, uu_n, u_1v_1, u_2v_2, \dots, u_nv_n\} \mid V[S(S_n)] \mid = 2n + 1, \mid E[S(S_n)] \mid = 2n$$

Define $f: E(G) \rightarrow \{1, 2\}$, for $i = 1, \dots, n$:

$$f(uu_i) = f(u_i v_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

Find $S = \sum_{uv \in E(G)} f(uv)$ and $P = \prod_{uv \in E(G)} f(uv)$. for each vertex in $V(S(S_n))$

Define $f^*: V(G) \rightarrow \{0, 1\}$ defined by $f(u) = \left\lfloor \frac{S^2 + P^2}{S + P} \right\rfloor \pmod{2}$

The obtained label values for the Subdivision of Star graph are summarized in Table 9.

	$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
n is even, $f^*(u) = 0$	$n+1$	n	n	n
n is even, $f^*(u) = 1$	n	$n + 1$	n	n

Table 9. Distribution of labels for Subdivision of Star graph $S(S_n)$

In each case the labeling defined as above satisfies $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(1) - e_f(2)| \leq 1$. Hence Subdivision of star graph $S(S_n)$ admits quotient square sum SP E-cordial labeling. That is Subdivision of star graph $S(S_n)$ is a quotient square sum SP E-cordial graph.

Theorem 10.

The Count Tree graph $CT_{n,n}$ admits a quotient square sum SP E-cordial labeling for every even $n \geq 6$.

Proof:

Let $V(CT_{n,n}) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ vertices and

$E(CT_{n,n}) = \{u_1u_2, u_2u_3, \dots, u_{n-1}u_n, u_nv_1, u_nv_2, \dots, u_nv_n\}$ edges

Here u_n is the support vertex and $v_1, \dots, v_n$ are pendant vertices attached to it.

Define $f: E(G) \rightarrow \{1, 2\}$:

$$f(u_i u_{i+1}) = 2 \ (i = 1, \dots, n-1), f(u_n v_i) = 1 \ (i = 1, \dots, n)$$

Find $S = \sum_{uv \in E(G)} f(uv)$ and $P = \prod_{uv \in E(G)} f(uv)$. for each vertex in $V(S(S_n))$

Define $f^*: V(G) \rightarrow \{0,1\}$ defined by $f(u) = \lfloor \frac{S^2+P^2}{S+P} \rfloor \pmod{2}$

The corresponding balanced labels for the Count Tree graph are displayed in Table 10.

	$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
If n is even	n	n	n	$n-1$

Table 10. Distribution of labels for Count Tree graph $CT_{n,n}$

In each case the labeling defined as above satisfies $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(1) - e_f(2)| \leq 1$. Hence Count Tree graph $CT_{n,n}$ admits quotient square sum SP E-cordial labeling. That is Count Tree graph $CT_{n,n}$ is a quotient square sum SP E-cordial graph.

Theorem 11.

The tadpole graph $T_{n,n}$ (also known as the dragon graph) admits a quotient square sum SP E-cordial labelling.

Proof:

Let P_n be a path with vertices $u_1, u_2, \dots, u_n$ and let the cycle edge $u_n u_1$ be added to close it into a cycle C_n . Attach a pendant path (tail) $v_1, v_2, \dots, v_{n-1}$ to the cycle by joining u_1 to v_1 with a bridge edge.

Let $V(T_{n,n}) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_{n-1}\}$ vertices and

Define $f: E(G) \rightarrow \{1,2\}$:

$$f(u_i u_{i+1}) = 1 \ (i = 1, \dots, n-1), f(u_n u_1) = 1, f(u_1 v_1) = 2, f(v_i v_{i+1}) = 2, 1 \leq i \leq n-2$$

Find $S = \sum_{uv \in E(G)} f(uv)$ and $P = \prod_{uv \in E(G)} f(uv)$. for each vertex in $V(T_{n,n})$

Define $f^*: V(G) \rightarrow \{0,1\}$ defined by $f(u) = \lfloor \frac{S^2+P^2}{S+P} \rfloor \pmod{2}$

The computed label distribution for the tadpole graph is given in Table 11.

$v_f(0)$	$v_f(1)$	$e_f(1)$	$e_f(2)$
$n-1$	n	n	$n-1$

Table 11. Vertex-edge label distribution for tadpole graph $T_{n,n}$

In each case the labeling defined as above satisfies $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(1) - e_f(2)| \leq 1$. Hence tadpole graph $T_{n,n}$ admits quotient square sum SP E-cordial labeling. That is tadpole graph $T_{n,n}$ is a quotient square sum SP E-cordial graph.

Conclusion and Open Problem:

In this paper, we introduced the Quotient Square Sum SP E-Cordial Labeling and established its existence for the path P_n , cycle C_n , comb graph $P_n \odot K_1$, crown graph $C_n \odot K_1$, triangular snake nC_3 , and the tadpole (dragon) graph $T_{n,n}$. These results confirm that the labeling scheme is well-defined and yields balanced vertex and

edge partitions across a variety of structurally distinct graph families — trees, unicyclic graphs, and graphs built from triangle chains. We additionally established the existence of this labeling for the helm graph, closed helm graph, semi-total point graph of a path, semi-total point graph of a cycle, subdivision of a star graph, and the count tree graph, for a total of twelve graph families treated in this paper. The following problems remain open and offer natural directions for further research:

Extension to other graph families. The existence of Quotient Square Sum SP E-Cordial labelings remains unexplored for several standard families, including the wheel graph W_n , fan graph F_n , gear graph G_n , book graph B_n , and star-related graphs such as the bistar $B(n)$, double star, and caterpillar $C(k)$. Establishing or disproving the labeling for these families would extend the scope of the theory considerably.

Extension to disconnected and product graphs. It would be of interest to study whether Quotient Square Sum SP E-Cordial labelings extend naturally to disjoint unions of graphs (e.g., $G_1 \cup G_2$), and to graph products such as the Cartesian product $G_1 \times G_2$ or the corona product $G_1 \odot G_2$ for general G_1, G_2 beyond the specific comb and crown cases treated here.

Author Contributions

P. Sujitha contributed to conceptualization, formal analysis, proof development, manuscript preparation, and writing of the original draft. Dr. S. Yahya Mohamed contributed to supervision, review, critical revision, and final approval of the manuscript. Both authors approved the final version of the manuscript and agreed to be accountable for all aspects of the work.

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