

On the various Connected Edge Dominations of Involutory Cayley Graph

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Abstract: In this paper, three edge domination parameters, namely connected edge domination, cototal edge domination, and complete cototal edge domination for an involutory Cayley graphs were presented. Theoretical results and derive domination numbers for involutory Cayley graphs of various orders were established. The structural properties of these graphs are utilized to construct suitable edge dominating sets and determine their minimum cardinalities. The obtained results provide a systematic characterization of these domination parameters for different classes of involutory Cayley graphs. These findings further contribute to the study of domination theory in algebraically structured graphs.

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1. Introduction

An involutory Cayley graph is a special class of Cayley graph constructed from a finite group using a set of involutory generators, and it provides an interesting framework for investigating various domination parameters. The concept of domination in graphs was introduced systematically by Cockayne and Hedetniemi [4], and subsequently several variants of domination have been developed. In particular, connected domination requires the subgraph induced by a dominating set to be connected. The connected domination number was studied by Sampathkumar and Walikar [13,14], while Laskar and Hedetniemi [9] further developed the theory of connected domination. For edge domination, Kulli and Sigarkanti [8] introduced the connected edge domination number, and Arumugam and Velammal [2] investigated connected edge domination in graphs. Another important variant is cototal domination, introduced and studied by Kulli, Janakiram and Iyer [6], in which the complement of the induced subgraph on the dominating set satisfies the required domination condition. The corresponding concept of cototal edge domination was investigated by Anupama, Maralabhavi and Goudar [1], while complete cototal domination and complete cototal edge domination have been studied by Basavanagoud and Hosamani [3] and Maria Regila Baby and Uma Samundesvari [10], respectively.

Involutory Cayley graphs were introduced as a new class of Cayley graphs by Venkata Anusha and Siva Parvathi [15], who established several structural properties of these graphs. Subsequently, Prameela Rani, Siva Parvathi and Lakshmi [11, 12], investigated domination and domatic numbers of involutory Cayley graphs], followed by studies on connected and independent domination in these graphs.

For a positive integer n , the involutory Cayley graph $(Z_n I_v)$ is a graph in which V is the vertex set and I_v denotes the set of all involutory elements in Z_n . Then two vertices $a, b \in Z_n$ are adjacent if and only if either $a - b \in I_v$ or $b - a \in I_v$.

Motivated by these developments, the present study focuses on connected edge domination, cototal edge domination, and complete cototal edge domination in involutory Cayley graphs for various cases.

2. Connected Edge Domination

In addition to vertex-based domination parameters, **edge domination** and its connected variants have received considerable attention. A set $D \subseteq E(G)$ is called a **connected edge dominating set** of a graph G if D' is an edge dominating set and the subgraph induced by the edges in D' is connected. The minimum cardinality of such a set is called the **connected edge domination number**, denoted by $\gamma'_c(G)$. The concept of connected edge



domination was introduced by Kulli and Sigarkanti [8], and subsequently studied in detail by Arumugam and Velammal [2]. This parameter provides a natural extension of connected domination from vertices to edges and is particularly relevant for studying the structural and domination properties of highly symmetric graphs such as involutory Cayley graphs. In this section, connected edge domination for the involutory Cayley graphs were discussed and presented results for various values of n of $G(Z_n, I_v)$.

Theorem 2.1: For $n = p^\alpha$, where p is odd prime, $\alpha \geq 1$, the connected edge domination number is

$$\gamma'_c(G(Z_n, I_v)) = n - 2.$$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{v_0, v_1, \dots, v_{n-1}\}$ and an edge set $E = \{e_{i,i+1} : 0 \leq i \leq n-2\} \cup \{e_{0,n-1}\}$. Then $|E| = n$.

Let $n = p^\alpha$, and p is odd prime, $\alpha \geq 1$.

Then $I_v = \{1, n-1\}$. The graph $G(Z_n, I_v)$ is a cycle and the adjacency of each edge is two.

Denote the subset of E , as $D' = \{e_{i,i+1} : 0 \leq i \leq n-3\}$, and $|D'| = n-2$.

Every edge $e_{i,i+1}$ of D' is adjacent to $e_{i-1,i}$ and $e_{i+1,i+2}$.

The first edge $e_{0,1}$ is adjacent to $e_{1,2}$ and $e_{n-1,0}$ and the last edge $e_{n-3,n-2}$ is adjacent to $e_{n-4,n-3}$ and $e_{n-2,n-1}$. Now the edges of D' dominates every edge of $G(Z_n, I_v)$ and there is a path between them.

Therefore D' is a connected edge dominating set with minimum cardinality.

Therefore, the connected edge domination number $\gamma'_c(G(Z_n, I_v)) = n - 2$.

Theorem 2.2: For $n = 2p^\alpha$, where p is odd prime, $\alpha \geq 1$, the connected edge domination number is

$$\gamma'_c(G(Z_n, I_v)) = n - 2.$$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{v_0, v_1, \dots, v_{n-1}\}$ and an edge set $E = \{e_{i,i+1} : 0 \leq i \leq n-2\} \cup \{e_{0,n-1}\}$. Then $|E| = n$.

Let $n = 2p^\alpha$, and p is odd prime, $\alpha \geq 1$. Then $I_v = \{1, n-1\}$.

The graph $G(Z_n, I_v)$ is a cycle and the adjacency of each edge is two.

Denote the subset of E , as $D' = \{e_{i,i+1} : 0 \leq i \leq n-3\}$ and $|D'| = n-2$.

Every edge $e_{i,i+1}$ of D' is adjacent to $e_{i-1,i}$ and $e_{i+1,i+2}$.

The first edge $e_{0,1}$ is adjacent to $e_{1,2}$ and $e_{n-1,0}$ and the last edge $e_{n-3,n-2}$ is adjacent to $e_{n-4,n-3}$ and $e_{n-2,n-1}$. Every edge of $E - D'$ has at least one end point incident with an edge of D' .

Hence every edge outside of D' is adjacent to an edge of D' .

Now the edges of D' dominates every edge of $G(Z_n, I_v)$ and there is a path between them. Therefore D' is a connected edge dominating set with minimum cardinality.

Therefore, the connected edge domination number $\gamma'_c(G(Z_n, I_v)) = n - 2$.

Theorem 2.3: For $n = 2^\alpha$, where $\alpha \geq 1$, the connected edge domination number is

$$\gamma'_c(G(Z_n, I_v)) = \frac{3n}{4} - 2.$$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{v_0, v_1, \dots, v_{n-1}\}$ and an edge set

$E = \{e_{i,k}: i \in Z_n, s \in I_v, k = (i + s)(\text{mod } n)\}$. Then $|E| = 2n$.

Let $n = 2^\alpha, \alpha \geq 1$. Then, $I_v = \{1, 2^{\alpha-1} - 1, 2^{\alpha-1} + 1, n - 1\}$.

The graph $G(Z_n, I_v)$ is a 4-regular and the adjacency of each edge is six.

Denote the subsets of E , as

$$D_1 = \{e_{0,s}: s \in I_v\},$$

$$D_2 = \{e_{i,i+1}: 1 \leq i \leq \frac{n}{2} - 4\}$$

$$D_3 = \{e_{2k,2k+l}: 1 \leq k \leq \frac{n}{4} - 2, l = 2^{\alpha-1} + 1\} \quad \text{and}$$

$$D_4 = \{\text{remaining all edges}\}.$$

$$\text{Then } |D_1| = 4, |D_2| = \frac{n}{2} - 4, |D_3| = \frac{n}{4} - 2, |D_4| = \frac{5n}{4} + 2.$$

Every edge of D_1 is adjacent to the remaining three edges of D_1 , and three edges of $D_2 \cup D_4$.

In D_2 , every edge is adjacent to the edges of $D_2 \cup D_3 \cup D_4$. In D_3 , every edge is adjacent to the edges of D_4 only.

$$\text{Let } D' = D_1 \cup D_2 \cup D_3.$$

Now the edges of D' dominates every edge of $G(Z_n, I_v)$ and there is a path between them.

Therefore D' is a connected edge dominating set with minimum cardinality.

$$\text{And } |D'| = |D_1 \cup D_2 \cup D_3| = |D_1| + |D_2| + |D_3| = 4 + \frac{n}{2} - 4 + \frac{n}{4} - 2 = \frac{3n+8}{4} = \frac{3n}{4} + 2.$$

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Therefore, the connected edge domination number $\gamma'_c(G(Z_n, I_v)) = \frac{3n}{4} - 2$.

Theorem 2.4: For $n = 2^2 p^\alpha$, where $\alpha \geq 1$, the connected edge domination number is

$$\gamma'_c(G(Z_n, I_v)) = \frac{3n}{4} - 2.$$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{v_0, v_1, \dots, v_{n-1}\}$ and an edge set

$$E = \{e_{i,k}: i \in Z_n, s \in I_v, k = (i + s)(\text{mod } n)\}. \text{ Then } |E| = 2n.$$

Let $n = 2^2 p^\alpha, \alpha \geq 1$. Then, $I_v = \{1, 2p^\alpha - 1, 2p^\alpha + 1, n - 1\}$.

The graph $G(Z_n, I_v)$ is a 4-regular and the adjacency of each edge is six.

Denote the subsets of E , as

$$D_1 = \{e_{0,s}: s \in I_v\},$$

$$D_2 = \{e_{i,i+1}: 1 \leq i \leq \frac{n}{2} - 4\}$$

$$D_3 = \{e_{2k,2k+l}: 1 \leq k \leq \frac{n}{4} - 2, l = 2p^\alpha + 1\} \quad \text{and}$$

$$D_4 = \{\text{remaining all edges}\}.$$

$$\text{Then } |D_1| = 4, |D_2| = \frac{n}{2} - 4, |D_3| = \frac{n}{4} - 2, |D_4| = \frac{5n}{4} + 2.$$

Every edge of D_1 is adjacent to the remaining three edges of D_1 , and three edges of $D_2 \cup D_4$.

In D_2 , every edge is adjacent to the edges of $D_2 \cup D_3 \cup D_4$. In D_3 , every edge is adjacent to the edges of D_4 only.

$$\text{Let } D' = D_1 \cup D_2 \cup D_3.$$

Now the edges of D' dominates every edge of $G(Z_n, I_v)$ and there is a path between them.

Therefore D' is a connected edge dominating set with minimum cardinality.

$$\text{And } |D'| = |D_1 \cup D_2 \cup D_3| = |D_1| + |D_2| + |D_3| = 4 + \frac{n}{2} - 4 + \frac{n}{4} - 2 = \frac{3n+8}{4} = \frac{3n}{4} + 2$$

$$\text{Therefore, the connected edge domination number } \gamma'_c(G(Z_n, I_v)) = \frac{3n}{4} - 2.$$

Theorem 2.5: For $n = 2^3 p^\alpha$, $\alpha \geq 1$, the connected edge domination number is

$$\gamma'_c(G(Z_n, I_v)) = \frac{5n}{8} - 2 \text{ (or) } 5p^\alpha - 2.$$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{v_0, v_1, \dots, v_{n-1}\}$ and an edge set $E = \{e_{i,k} : i \in Z_n, s \in I_v, k = (i + s)(\text{mod } n)\}$. Then $|E| = 4n$.

Let $n = 2^3 p^\alpha$, where $\alpha \geq 1$. Then, $I_v = \{1, 2p^\alpha - 1, 2p^\alpha + 1, 2^2 p^\alpha - 1, 2^2 p^\alpha + 1,$

$$2^3 p^\alpha - 2p^\alpha - 1, 2^3 p^\alpha - 2p^\alpha + 1, n - 1\}.$$

The graph $G(Z_n, I_v)$ is a 8-regular graph and the adjacency of each edge is fourteen.

Denote the subset of E , as

$$D_1 = \{e_{0,s} : s \in I_v\},$$

$$D_2 = \{e_{i,i+1} : 1 \leq i \leq \frac{n}{4} - 4\},$$

$$D_3 = \{e_{2k,2k+l} : 1 \leq k \leq \frac{n}{8} - 2, l \in \{2p^\alpha + 1, 2^2 p^\alpha + 1, 2^3 p^\alpha - 2p^\alpha + 1\}\} \text{ and}$$

$$D_4 = \{\text{remaining all edges}\}.$$

$$\text{Then } |D_1| = 8, |D_2| = \frac{n}{4} - 4, |D_3| = 3(\frac{n}{8} - 2), |D_4| = \frac{27n}{8} + 2.$$

Every edge of D_1 is adjacent to the remaining seven edges of D_1 , and seven edges of $D_2 \cup D_4$.

In D_2 , every edge is adjacent to the edges of $D_2 \cup D_3 \cup D_4$. In D_3 , every edge is adjacent to the edges of D_4 only.

$$\text{Let } D' = D_1 \cup D_2 \cup D_3.$$

Now the edges of D' dominates every edge of $G(Z_n, I_v)$ and there is a path between them.

Therefore D' is a connected edge dominating set with minimum cardinality.

$$\text{And } |D'| = |D_1 \cup D_2 \cup D_3| = |D_1| + |D_2| + |D_3| = 8 + \frac{n}{4} - 4 + 3(\frac{n}{8} - 2) = \frac{5n}{8} - 2$$

$$\text{Therefore, the connected edge domination number } \gamma'_c(G(Z_n, I_v)) = 5p - 2.$$

3. COTOTAL EDGE DOMINATION

The concept of cototal domination was introduced and studied by Kulli, Janakiram and Iyer [6]. Further developments in this direction include cototal edge domination, investigated by Anupama, Maralabhavi and Goudar [1]. In this section, the **cototal edge domination** of an Involutory Cayley graph was presented and established results for various values of n of $G(Z_n, I_v)$.

Let $G(V, E)$ be a graph and a dominating set $D' \subseteq E(G)$ is called a **cototaledgedominating set** if the subgraph D' is dominated and $\langle E - D' \rangle$ has no isolated edges. The minimum cardinality of a cototal edge dominating set is called the **cototal edge domination number**, denoted by $\gamma'_{cot}(G)$.

Theorem 3.1: For the involutory Cayley graph $G(Z_n, I_v)$, when $n = P^\alpha$, where p is an odd prime and $p \neq 3$ and $\alpha \geq 1$, the co-total edge domination number is

$$\gamma'_{cot}(G(Z_n, I_v)) = \left\{ \frac{n+2}{3} \text{ if } n \equiv 1(\text{mod } 3) \quad \frac{n+4}{3} \text{ if } n \equiv 2(\text{mod } 3) \right.$$

Proof: consider the graph $G(Z_n, I_v)$ with vertex set $V = \{v_0, v_1 \dots \dots v_{n-1}\}$ and an edge set

$$E = \{e_{i,i+1} : 0 \leq i \leq n-2\} \cup \{e_{0,n-1}\}. \text{ Then } |E| = n.$$

Let $n = p^\alpha$, $p \neq 3$, $\alpha \geq 1$. Then $I_v = \{1, n-1\}$. The graph $G(Z_n, I_v)$ is a cycle and the adjacency of each edge is two. Here two cases arise,

Case 1: $n \equiv 1(\text{mod } 3)$.

Denote the subset of E , as $D' = \{e_{3k,3k+1} : 0 \leq k \leq \frac{n-1}{3}\}$ and $|D'| = \frac{n+2}{3}$.

Every edge $e_{3k,3k+1}$ of D' is adjacent to $e_{3k-1,3k}$ and $e_{3k+1,3k+2}$.

And the last edge $e_{\frac{n-1}{3},0}$ is adjacent to $e_{0,1}$ and $e_{n-4,n-3}$.

Now the edges of D' dominates every edge of $G(Z_n, I_v)$ and the induced subgraph $\langle E - D' \rangle$ has no isolated edges. Therefore D' is co-total edge dominating set with minimum cardinality.

Therefore, the co-total edge domination number $\gamma'_{cot}(G(Z_n, I_v)) = \frac{n+2}{3}$.

Case (ii): Let $n \equiv 2(\text{mod } 3)$.

Represent the subset of E , as $D' = \{e_{3k,3k+1} : 0 \leq k \leq \frac{n-2}{3}\} \cup \{e_{n-1,0}\}$, so $|D'| = \frac{n+4}{3}$.

Every edge $e_{3k,3k+1}$ of D' is adjacent to $e_{3k-1,3k}$ and $e_{3k+1,3k+2}$.

Now the edges of D' dominates each edge of $G(Z_n, I_v)$ and the induced subgraph $\langle E - D' \rangle$ has no independent edges. Therefore D' is co-total edge dominating set with minimum order.

And $|D'| = \frac{n+1}{3} + 1 = \frac{n+4}{3}$. Therefore, the co-total edge domination number of $G(Z_n, I_v) = \gamma'_{cot}(G(Z_n, I_v)) = \frac{n+4}{3}$.

Theorem 3.2: For the involutory Cayley graph $G(Z_n, I_v)$, when $n = 2P^\alpha$, where p is an odd prime, where $\alpha \geq 1$, the co-total edge domination number is

$$\gamma'_{cot}(G(Z_n, I_v)) = \left\{ \frac{n}{3} \text{ if } n \equiv 0(\text{mod } 3) \quad \frac{n+2}{3} \text{ if } n \equiv 1(\text{mod } 3) \quad \frac{n+4}{3} \text{ if } n \right. \\ \left. \equiv 2(\text{mod } 3) \right.$$

Proof: consider the graph $G(Z_n, I_v)$ with vertex set $V = \{v_0, v_1 \dots \dots v_{n-1}\}$ and an edge set

$$E = \{e_{i,i+1} : 0 \leq i \leq n-2\} \cup \{e_{0,n-1}\}. \text{ Then } |E| = n.$$

Let $n = 2p^\alpha$, $\alpha \geq 1$. Then $I_v = \{1, n-1\}$. The graph $G(Z_n, I_v)$ is a circular graph and the adjacency of each edge is two. Here three cases arise,

Case 1: $n \equiv 0(\text{mod } 3)$.

Denote the subset of E , as $D' = \{e_{3k,3k+1} : 0 \leq k \leq \frac{n-3}{3}\}$.and $|D'| = \frac{n}{3}$.

Every edge $e_{3k,3k+1}$ of D' is adjacent to $e_{3k-1,3k}$ and $e_{3k+1,3k+2}$. Every edge of $G(Z_n, I_v)$, not belonging to D' is adjacent to atleast one edge in D' and no edge remains isolated in the induced subgraph $\langle E - D' \rangle$. Therefore D' is co-total edge dominating set with minimum order.

Therefore, the co-total edge domination number $\gamma'_{cot}(G(Z_n, I_v)) = \frac{n}{3}$.

Case 2: $n \equiv 1 \pmod{3}$.

Denote the subset of E , as $D' = \{e_{3k,3k+1} : 0 \leq k \leq \frac{n-1}{3}\}$.and $|D'| = \frac{n+2}{3}$.

Each edge $e_{3k,3k+1}$ of D' is incident with the edges $e_{3k-1,3k}$ and $e_{3k+1,3k+2}$.

Now the edges of D' dominates each edge of $G(Z_n, I_v)$ and the induced subgraph $\langle E - D' \rangle$ has no independent edges. Therefore D' is co-total edge dominating set with minimum order.

Therefore, the co-total edge domination number $\gamma'_{cot}(G(Z_n, I_v)) = \frac{n+2}{3}$.

Case (ii): Let $n \equiv 2 \pmod{3}$.

Represent the subset of E , as $D' = \{e_{3k,3k+1} : 0 \leq k \leq \frac{n-2}{3}\} \cup \{e_{n-1,0}\}$, so $|D'| = \frac{n+4}{3}$.

Every edge $e_{3k,3k+1}$ of D' is adjacent to $e_{3k-1,3k}$ and $e_{3k+1,3k+2}$.

Moreover, the edges $e_{n-1,0}$ is incident with $e_{0,1}$ and $e_{n-2,n-1}$.

Now the edges of D' dominates each edge of $G(Z_n, I_v)$ and the induced subgraph $\langle E - D' \rangle$ has no independent edges. Therefore D' is co-total dominating set with minimum cardinality.

And $|D'| = \frac{n+1}{3} + 1 = \frac{n+4}{3}$. Therefore, the co-total edge domination number of $G(Z_n, I_v) = \gamma'_{cot}(G(Z_n, I_v)) = \frac{n+4}{3}$.

Theorem 3.3: For the involutory Cayley graph $G(Z_n, I_v)$ when $n = 2^\alpha$, where $\alpha \geq 3$, the co-total edge domination number is

$$\gamma'_{cot}(G(Z_n, I_v)) = \left\{ \begin{array}{ll} \frac{n+2}{3} & \text{if } n \equiv 1 \pmod{3} \\ \frac{n+4}{3} & \text{if } n \equiv 2 \pmod{3} \end{array} \right.$$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $\{v_0, v_1, \dots, v_{n-1}\}$ and edge set

$$E = \{e_{i,k} : i \in Z_n, s \in I_v, k = (i + s) \pmod{n}\}. \text{ Then } |E| = 2n.$$

Let $n = 2^\alpha$, where $\alpha \geq 3$. Then, $I_v = \{1, 2^{\alpha-1} - 1, 2^{\alpha-1} + 1, n - 1\}$.

Case (i): $n \equiv 1 \pmod{3}$.

Choose two subsets D_1 and D_2 of E such that

$$D_1 = \{e_{ij} : i \equiv 0 \pmod{3}, 0 \leq i \leq \frac{n}{2} + 1\} \text{ and } |D_1| = \frac{n+8}{6},$$

$$D_2 = \{e_{ij} : i \equiv 2 \pmod{3}, \frac{n}{2} + 3 \leq i \leq n - 2\} \text{ and } |D_2| = \frac{n-4}{6} \text{ and}$$

$$D_3 = \{\text{remaining edges of } G\}.$$

Every edge of D_1 is adjacent to D_3 , and every edge of D_2 is also adjacent to D_3 .

$$\text{Let } D' = D_1 \cup D_2.$$

Now the edges of D' dominates each edge of $G(Z_n, I_v)$ and the induced subgraph $\langle E - D' \rangle$ has no independent edges. Therefore D' is co-total dominating set with minimum cardinality.

$$\text{And } |D'| = \frac{n+8}{6} + \frac{n-4}{6} = \frac{2n+4}{6} = \frac{n+2}{3}.$$

Therefore, the co-total edge domination number of $G(Z_n, I_v) = \gamma'_{cot}(G(Z_n, I_v)) = \frac{n+2}{3}$.

Case (ii): $n \equiv 2 \pmod{3}$.

Choose two subsets D_1 and D_2 of E such that:

$$D_1 = \{e_{ij} : i \equiv 0 \pmod{3}, 0 \leq i \leq \frac{n}{2} - 1\}, \text{ and } |D_1| = \frac{n+4}{6}$$

$$D_2 = \{e_{ij} : i \equiv 1 \pmod{3}, \frac{n}{2} \leq i \leq n - 1\} \text{ and } |D_2| = \frac{n+4}{6}$$

$$D_3 = \{\text{remaining edges of } G\}.$$

The first and last edges of D_1 are adjacent to $D_2 \cup D_3$ and the remaining edges are adjacent to D_3 .

Similarly except first and last edges of D_2 are adjacent to D_3 .

$$\text{Let } D' = D_1 \cup D_2.$$

Now the edges of D' dominates each edge of $G(Z_n, I_v)$ and the induced subgraph $\langle E - D' \rangle$ has no independent edges. Therefore D' is co-total dominating set with minimum cardinality.

$$\text{And } |D'| = \frac{n+4}{6} + \frac{n+4}{6} = \frac{2n+8}{6} = \frac{n+4}{3}.$$

Therefore, the co-total edge domination number of $G(Z_n, I_v) = \gamma'_{cot}(G(Z_n, I_v)) = \frac{n+4}{3}$.

Theorem 3.4: For the involutory Cayley graph $G(Z_n, I_v)$ when $n = 2^2 p^\alpha$, where p is an odd prime and

$\alpha \geq 1$, the co-total domination number is

$$\gamma'_{cot}(G(Z_n, I_v)) = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod{3} \\ \frac{n+2}{3} & \text{if } n \equiv 1 \pmod{3} \\ \frac{n+4}{3} & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{v_0, v_1, \dots, v_{n-1}\}$ and edge set

$$E = \{e_{i,k} : i \in Z_n, s \in I_v, k = (i + s) \pmod{n}\}. \text{ Then } |E| = 2n.$$

Let $n = 2^2 p^\alpha$, where p is an odd prime and $\alpha \geq 1$. Then, $I_v = \{1, 2p^\alpha - 1, 2p^\alpha + 1, n - 1\}$.

The graph $G(Z_n, I_v)$ is a 4-regular and the adjacency of each edge is six.

Case 1: Suppose $n \equiv 0 \pmod{3}$.

$$\text{Denote the subset of } E, \text{ as } D' = \{e_{3k, 3k+1} : 0 \leq k \leq \frac{n-3}{3}\} \text{ and } |D'| = \frac{n}{3}.$$

Every edge $e_{3k, 3k+1}$ of D' is adjacent to $e_{3k-1, 3k}$ and $e_{3k+1, 3k+2}$.

Every edge of $G(Z_n, I_v)$, not belonging to D' is adjacent to at least one edge in D' and no edge remains isolated in the induced subgraph $\langle E - D' \rangle$. Therefore D' is co-total edge dominating set with minimum cardinality.

$$\text{Therefore, the co-total edge domination number } \gamma'_{cot}(G(Z_n, I_v)) = \frac{n}{3}.$$

Case 2: $n \equiv 1(mod\ 3)$

Denote the subset of E , as $D' = \{e_{3k,3k+1} : 0 \leq k \leq \frac{n-1}{3}\} \cup \{e_{n-3,n-2}\}$. and $|D'| = \frac{n+2}{3}$.

Each edge $e_{3k,3k+1}$ of D' is incident with the edges $e_{3k-1,3k}$ and $e_{3k+1,3k+2}$.

$e_{n-3,n-2}$ is adjacent to $e_{n-4,n-3}$ and $e_{n-2,n-1}$

Now the edges of D' dominates each edge of $G(Z_n, I_v)$ and the induced subgraph $\langle E - D' \rangle$ has no independent edges. Therefore D' is co-total edge dominating set with minimum order.

Therefore, the co-total edge domination number $\gamma'_{cot}(G(Z_n, I_v)) = \frac{n+2}{3}$.

Case 3: Let $n \equiv 2(mod\ 3)$.

Represent the subset of E , as

$$D' = \left\{e_{3k,3k+1} : 0 \leq k \leq \frac{n-2}{3}\right\} \cup \{e_{n-1,0}\}, \quad \text{so } |D'| = \frac{n+4}{3}.$$

Every edge $e_{3k,3k+1}$ of D' is adjacent to $e_{3k-1,3k}$ and $e_{3k+1,3k+2}$.

And also, the edge $e_{n-1,0}$ is incident with $e_{0,1}$ and $e_{n-2,n-1}$.

Now the edges of D' dominates each edge of $G(Z_n, I_v)$ and the induced subgraph $\langle E - D' \rangle$ has no isolated edges. Therefore D' is co-total dominating set with minimum cardinality.

$$\text{And } |D'| = \frac{n+1}{3} + 1 = \frac{n+4}{3}.$$

Therefore, the co-total edge domination number of $G(Z_n, I_v) = \gamma'_{cot}(G(Z_n, I_v)) = \frac{n+4}{3}$.

Theorem.3.5: For the involutory Cayley graph $G(Z_n, I_v)$ when $n = 3P$, where P is an odd prime, the co-total edge domination number $\gamma'_{cot}(G(Z_n, I_v)) = P$

Proof: Consider the graph (Z_n, I_v) with vertex set $\{v_0, v_1 \dots v_{n-1}\}$ & edge set $E = \{e_{i,k} : i \in Z_n, s \in I_v, k = (i+s)(mod\ n)\}$. Then $|E| = 2n$,

Let $n = 3P$. Then, $I_v = \{1, p+1, 2p-1, n-1\}$ for $p \equiv 1(mod\ 3)$ & $\{1, p-1, 2p+1, n-1\}$ for $n \equiv 2(mod\ 3)$. Here $|I_v| = 4$.

Consider the subset of E , as $D' = \{e_{3k,3k+1} : 0 \leq k \leq \frac{n-3}{3}\}$ and $|D'| = \frac{n}{3}$.

D' is a dominating set, because if we remove any edge from D' the set D' no longer dominates all the edges. Therefore, D' is dominating set and $E - D'$ has no isolated edges. Therefore D' is a co-total edge dominating set with minimum cardinality P . Therefore, the co-total domination number is P .

4. COMPLETE COTOTAL EDGE DOMINATION

The concept of complete cototal edge domination combines the ideas of edge domination, cototal domination, and completeness. The study of cototal edge domination was developed by Anupama, Maralabhavi and Goudar [1], while complete cototal edge domination was investigated by Maria Regila Baby and Uma Samundesvari [10]. These concepts, together with the earlier work on complete cototal domination by Basavanagoud and Hosamani [3].

A set $D' \subseteq E(G)$ is called a complete cototal edge dominating set if D' is a cototal edge dominating set and the edges in D' satisfy the completeness condition. The minimum cardinality of such a set is called the **complete** cototal edge domination number, denoted by $\gamma'_{ccot}(G)$.

In this section, the complete cototaledge domination of $G(Z_n, I_v)$ is discussed and presented some result for various values of n .

Theorem 4.1: For the involutory Cayley graph $G(Z_n, I_v)$ when $n = P^\alpha$, where p is an odd prime and $\alpha \geq 1$, the complete co-total domination number is

$$\gamma'_{ccot}(G(Z_n, I_v)) = \begin{cases} \frac{n+1}{2} & \text{if } n \equiv 1 \pmod{4} \\ \frac{n+3}{2} & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{v_0, v_1, \dots, v_{n-1}\}$ and an edge set

$$E = \{e_{i,i+1} : 0 \leq i \leq n-2\} \cup \{e_{0,n-1}\}. \text{ Then } |E| = n.$$

Let $n = P^\alpha$, where p is an odd prime, $\alpha \geq 1$. Then $I_v = \{1, n-1\}$.

The graph $G(Z_n, I_v)$ is a cycle and the adjacency of each edge is two.

Here two possible cases, since n is odd, either $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$.

Case 1: $n \equiv 1 \pmod{4}$.

Define the subset of the edge set E , as $D' = \{e_{ij} : i \equiv 0, 1 \pmod{4}, 0 \leq i \leq n-1, j = i+1\}$.

$$\text{Then } |D'| = \frac{n+1}{2}.$$

D' is a total dominating set of $G(Z_n, I_v)$ and every edge in $E - D'$ has an adjacent edge in $E - D'$, so the induced subgraph $\langle E - D' \rangle$ has no isolated edges in $G(Z_n, I_v)$. Therefore D' is a complete co-total edge domination set, and $|D'| = \frac{n+1}{2}$. Hence $\gamma'_{ccot}(G(Z_n, I_v)) = \frac{n+1}{2}$.

Case 2: $n \equiv 3 \pmod{4}$.

Define the subset of E , as $D' = \{e_{ij} : i \equiv 0, 1 \pmod{4}, j = i+1\} \cup \{e_{0,n-1}\}$.

$$\text{Then } |D'| = \frac{n+3}{2}$$

D' is a total dominating set of $G(Z_n, I_v)$ and every edge in $E - D'$ has an adjacent edge in $E - D'$, so the induced subgraph $\langle E - D' \rangle$ has no isolated edges in $G(Z_n, I_v)$. Therefore D' is a complete co-total edge domination set with minimum cardinality.

$$\text{Therefore, the complete co-total edge domination number } \gamma'_{ccot}(G(Z_n, I_v)) = \frac{n+3}{2}.$$

Theorem 4.2: For the involutory Cayley graph $G(Z_n, I_v)$ when $n = 2P^\alpha$, where p is an odd prime and $\alpha \geq 1$, the complete cototal edge domination number $\gamma'_{ccot}(G(Z_n, I_v)) = \frac{n}{2} + 1$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $\{v_0, v_1, \dots, v_{n-1}\}$ and edge set

$$E = \{e_{i,i+1} : 0 \leq i \leq n-2\} \cup \{e_{0,n-1}\}. \text{ Then } |E| = n.$$

Let $n = 2P^\alpha$, where p is an odd prime, $\alpha \geq 1$.

Then $I_v = \{1, n-1\}$.

The graph $G(Z_n, I_v)$ is a circular graph and the adjacency of each edge is two.

Consider the subset of E , as

$$D' = \{e_{ij} : i \equiv 0, 1 \pmod{4}, j = i+1\}.$$

$$\text{Then } |D'| = \frac{n}{2} + 1$$

In D' every edge is adjacent to atleast one edge in D' , therefore D' is a total dominating set of $G(Z_n, I_v)$ and every edge in $E - D'$ has an adjacent edge in $E - D'$, so the induced subgraph $\langle E - D' \rangle$ has no isolated edges in $G(Z_n, I_v)$. Therefore D' is a complete co-total edge domination set with minimum cardinality.

Therefore, the complete co-total edge domination number $\gamma'_{ccot}(G(Z_n, I_v)) = \frac{n}{2} + 1$

Theorem 4.3: For the involutory Cayley graph $G(Z_n, I_v)$ when $n = 3P$, where P is prime, the complete co-total edge domination number is $\gamma'_{ccot}(G(Z_n, I_v)) = \frac{n+1}{2}$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $\{v_0, v_1 \dots \dots v_{n-1}\}$ and edge set $E = \{e_{i,k} : i \in Z_n, s \in I_v, k = (i + s)(\text{mod } n)\}$. Then $|E| = 2n$.

Let $n = 3P, P$ any prime. Then $I_v = \{1, p + 1, 2p - 1, n - 1\}$ for $p \equiv 1(\text{mod } 3)$ & $\{1, p - 1, 2p + 1, n - 1\}$ for $n \equiv 2(\text{mod } 3)$.

The graph is a 4-regular graph and the adjacency of each edge is six.

Here two cases arise,

Case 1: $p \equiv 1(\text{mod } 3)$

Then $I_v = \{1, p + 1, 2p - 1, n - 1\}$ consider the subset of E , as

$$D' = \{e_{ij} : i \equiv 0, 1(\text{mod } 4), j = i + 1\} \cup \{e_{0, n-1}\}.$$

$$\text{Then } |D'| = \frac{n+1}{2}.$$

D' have no isolated edges therefore D' is total dominating set, and $E - D'$ has no isolated edges, with $|D| = (n + 1)/2$.

Therefore D' is complete co-total edge dominating set with minimum cardinality

Therefore, the complete co-total edge domination number $\gamma'_{ccot}(G(Z_n, I_v)) = \frac{n+1}{2}$

Case 2: $p \equiv 2(\text{mod } 3)$

Then $I_v = \{1, p - 1, 2p + 1, n - 1\}$

consider the subset of E , as $D' = \{e_{ij} : i \equiv 0, 1(\text{mod } 4), j = i + 1\}$.

$$\text{Then } |D'| = \frac{n+1}{2}.$$

D' have no isolated edges therefore D' is total dominating set, and $E - D'$ has no independent edges, with $|D| = (n + 1)/2$. Therefore D' is complete co-total edge dominating set with minimum cardinality, the complete co-total edge domination number $\gamma'_{ccot}(G(Z_n, I_v)) = \frac{n+1}{2}$

Theorem 4.4: For the involutory Cayley graph $G(Z_n, I_v)$ when $n = 2^\alpha$, where $\alpha > 1$, the complete co-total edge domination number is $\gamma'_{ccot}(G(Z_n, I_v)) = n/2$.

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $\{v_0, v_1 \dots \dots v_{n-1}\}$ and edge set $E = \{e_{i,k} : i \in Z_n, s \in I_v, k = (i + s)(\text{mod } n)\}$. Then $|E| = 2n$.

Let $n = 2^\alpha, \alpha > 1$, Then, $I_v = \{1, 2^{\alpha-1} - 1, 2^{\alpha-1} + 1, n - 1\}$.

The graph $G(Z_n, I_v)$ is a 4-regular and the adjacency of each edge is six

Consider the subset E , as $D' = \{e_{ij} : i \equiv 0, 1(\text{mod } 4), j = i + 1\}$.

$$\text{Then } |D'| = \frac{n}{2}.$$

D' total edge dominating set, , and for every $e_{ij} \in E - D'$ there exists another edge which is adjacent to e_{ij} . Therefore D' is a complete co-total edge dominating set, with minimum cardinality.

$$\text{So, the complete co-total edge domination number } \gamma'_{ccot}(G(Z_n, I_v)) = \frac{n}{2}.$$

Theorem 4.5: For the involutory Cayley graph $G(Z_n, I_v)$ when $n = 2^2 p^\alpha$, where $\alpha > 1$, the complete co-total edge domination number is $\gamma'_{ccot}(G(Z_n, I_v)) = n/2$.

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $\{v_0, v_1 \dots \dots v_{n-1}\}$ and edge set $E = \{e_{i,k} : i \in Z_n, s \in I_v, k = (i + s)(\text{mod } n)\}$. Then $|E| = 2n$.

$$\text{Let } n = 2^2 p^\alpha, \alpha > 1, \text{ Then, } I_v = \{1, 2p^\alpha - 1, 2p^\alpha + 1, n - 1\}.$$

The graph $G(Z_n, I_v)$ is a 4-regular and the adjacency of each edge is six

$$\text{Consider the subset } E, \text{ as } D' = \{e_{ij} : i \equiv 0, 1 (\text{mod } 4), j = i + 1\}. \text{ Then } |D'| = \frac{n}{2}.$$

D' total edge dominating set, , and for every $e_{ij} \in E - D'$ there exists another edge which is adjacent to e_{ij} . Therefore D' is a complete co-total edge dominating set, with minimum cardinality.

$$\text{So, the complete co-total edge domination number } \gamma'_{ccot}(G(Z_n, I_v)) = \frac{n}{2}.$$

Theorem 4.6: For $n = 2^3 p^\alpha, \alpha \geq 1$, the complete co-total edge domination number is

$$\gamma'_c(G(Z_n, I_v)) = \frac{n}{2}.$$

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $V = \{v_0, v_1, \dots, v_{n-1}\}$ and an edge set $E = \{e_{i,k} : i \in Z_n, s \in I_v, k = (i + s)(\text{mod } n)\}$. Then $|E| = 4n$.

$$\text{Let } n = 2^3 p^\alpha, \text{ where } \alpha \geq 1.$$

$$\text{Then, } I_v = \{1, 2P^\alpha - 1, 2P^\alpha + 1, 2^2 P^\alpha - 1, 2^2 P^\alpha + 1, 2^3 P^\alpha - 2P^\alpha - 1, 2^3 P^\alpha - 2P^\alpha + 1, n - 1\}.$$

The graph $G(Z_n, I_v)$ is a 8-regular graph and the adjacency of each edge is fourteen.

Denote the subset of E , as

$$D' = \{e_{ij} : i \equiv 0, 1 (\text{mod } 4), j = i + 1\}.$$

$$\text{Then } |D'| = \frac{n}{2}.$$

Every edge in D' is adjacent to atleast one edge in D' . therefore D' is total edge dominating set, and for every $e_{ij} \in E - D'$ there exists another edge which is adjacent to e_{ij} .

Therefore D' is a complete co-total edge dominating set, with minimum cardinality.

$$\text{Hence, the complete co-total edge domination number } \gamma'_{ccot}(G(Z_n, I_v)) = \frac{n}{2}.$$

Theorem 4.7: For the involutory Cayley graph $G(Z_n, I_v)$, $n = P_1^\alpha P_2^\alpha (\alpha \geq 1)$, the complete co-total edge domination number is $\gamma'_{ccot}(G(Z_n, I_v)) = \frac{n+1}{2}$.

Proof: Consider the graph $G(Z_n, I_v)$ with vertex set $\{v_0, v_1 \dots \dots v_{n-1}\}$ and edge set $E = \{e_{i,k} : i \in Z_n, s \in I_v, k = (i + s)(\text{mod } n)\}$. Then $|E| = 4n$.

$$\text{Let } n = P_1^\alpha P_2^\alpha \dots \dots P_n^\alpha.$$

Then $I_v = \{I_{v1}, I_{v2}, I_{v3}, I_{v4}\}$, $I_{v1} < I_{v2} < I_{v3} < I_{v4}$.

The graph $G(Z_n, I_v)$ is a 4-regular and the adjacency of each edge is six

Consider the subset E , as $D' = \{e_{ij} : i \equiv 0, 1 \pmod{4}, j = i + 1\}$.

Then $|D'| = \frac{n+1}{2}$.

D' total edge dominating set, and for every $e_{ij} \in E - D'$ there exists another edge which is adjacent to e_{ij} . Therefore D' is a complete co-total edge dominating set, with minimum cardinality.

Hence, the complete co-total edge domination number $\gamma'_{ccot}(G(Z_n, I_v)) = \frac{n+1}{2}$.

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