International Journal of Computer Information Systems and Industrial Management Applications
ISSN 2150-7988 Volume 18(20s) 2026 pp. 1901-1909
https://doi.org/10.70917/ijcisim-2026-6113
Article

Semi-Analytic Solution of a Linear Space-Fractional Diffusion Equation by the Method of Lines

Jignasa Panchal[1]*, Gautam Patel[2]

[1]Department of Mathematics, Government Polytechnic, Himatnagar, Gujarat, India
[2]Department of Mathematics, Gandhinagar Institute of Science, Gandhinagar University, Gandhinagar, Gujarat, India
*Corresponding author: jignasapanchal14@gmail.com

Abstract: The study of space-fractional diffusion equations is important for modelling anomalous diffusion in physics, engineering, and biomedical sciences. In this work, a semi-analytic solution for a linear space-fractional diffusion equation is developed using the method of lines. The spatial fractional derivative is discretized using central finite differences, transforming the problem into a system of ordinary differential equations in time. This system is then solved using the variation of parameters method, yielding a semi-analytic solution. Numerical results and error analysis demonstrate the accuracy and stability of the proposed method, which can be extended to more complex fractional models.
Keywords: Space-fractional diffusion equation; Method of lines; Semi-analytic solution; Fractional calculus; Variation of parameter method

1. Introduction

Partial differential equations are widely used to describe many real-world problems in science and engineering, such as heat transfer, fluid flow and diffusion processes. Among these, advection–diffusion equations play an important role in modelling how substances move and spread over time. However, in many practical situations, these processes are affected by uncertainty and irregular behaviour, which cannot be fully explained by classical equations.

To better represent such complex systems, fractional calculus has been introduced. Fractional derivatives help capture memory and nonlocal effects, making the models more accurate. In addition, randomness can be included in the equations to account for uncertain factors, leading to fractional stochastic advection–diffusion equations. These types of equations are more realistic but also more difficult to solve exactly.

Because of this difficulty, numerical methods are commonly used to find approximate solutions. In this paper, we focus on developing a simple and efficient numerical approach based on finite difference and variation parameter method which is known as Method of Lines(MOL). The main idea is to convert the original problem into a form that can be solved more easily using variation parameter method. we present a semi-analytical method of lines (MOL) solution. The MOL were applied to solve the PDEs by Schiesser et al. [20]. This Schiesser's method is a technique based on fully numerical scheme. In 2004, Koto [21] applied this technique to approximations of delay di_erential equations using Runge-Kutta method. Hamdi et al. [22] gave basic idea of MOL. The semi-analytic MOL is used actively for solving linear PDEs. For example see [23, 24, 25].

The proposed method is tested for accuracy and reliability through numerical examples. The results show that the method performs well and provides solutions that are close to the exact values. This makes it a useful tool for solving complex fractional stochastic problems in practical applications [1-6].

Consider the one dimensional space fractional diffusion equation (SFDE) with source term of the form

$$\frac{\partial u\,(x,t)}{\partial t} = d(x,t)\ \frac{\partial^{\alpha} u\,(x,t)}{\partial x^{\alpha}} + p\left(x,t,u(x,t)\right) \quad , \quad (x,t)\,\epsilon\,\Omega \qquad (1)$$

where $1< \alpha \leq 2$, $\Omega = \{(x, t): a < x < b, t > 0\}$ and continuous function $d(x,t) > 0$ denotes the diffusion coefficient (or diffusivity).The forcing function p (x , t , u (x , t)) which is assumed to be continuous and satisfies the Lipschitz condition with respect to third argument, can be used to represent sources and sinks, and given as linear function in u (x , t) [1].

The initial condition of the SFDE (1) is given as

$$u\,(x, 0) = \varphi(x),\ a < x < b \qquad (2)$$

Robin boundary conditions (BCs) are

$$c_1(t)\,u(a,t) + \ c_2(t)\frac{\partial u(a,t)}{\partial x} = \ h_1(t) \qquad (3)$$

$$c_3(t)\,u(b,t) + \ c_4(t)\frac{\partial u(b,t)}{\partial x} = \ h_2(t) \qquad (4)$$

where $c_1(t)$, $c_2(t)$ and $c_3(t)$, $c_4(t)$ do not vanish simultaneously for $t > 0$ [1]

If $c_2(t) = c_4(t) = 0$, we have Dirichlet BCs

If $c_1(t) = c_3(t) = 0$, we have Neumann BCs.

Although in most practical applications $\{c_i(t)\}_{i=1}^{i=4}$ are given as constants, but here we consider the general case of variation with t [1].

For $p\left(x,t,u(x,t)\right) = p(x,t)$ Khader [7] used a Chebyshev allocation method to discretize Eq. (1) to get a linear system of ordinary differential equations (ODEs) and used finite differences (FDs) to solve the resulting ODE system. Zhu [8] studied SFDEs with fractional Fellers operator. Saadatmandi and Dehghan [9] introduced approximation techniques based on the shifted Legendre–tau (SLT) scheme. Ren et al. [10] presented a method based on the shifted Chebyshev-tau, and using the operational matrix of the fractional derivative, reduced the problem to a set of linear algebraic equations.Sousa [11] presented an implicit numerical method which used a spline approximation to approximate the fractional derivative.Here in our presentation we use central finite difference method to discretize second order derivative term in Caputo equation**.**

Caputo Fractional Derivative

In mathematics, the **Caputo fractional derivative**, also called Caputo-type fractional derivative, is a generalization of derivatives for non-integer orders named after Michele Caputo. Caputo first defined this form of fractional derivative in 1967 [12].

In Caputo sense

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = \ \frac{1}{\Gamma(2-\alpha)}\ \int_a^x (x-s)^{1-\alpha}\,.\ \frac{\partial^2 u(s,t)}{\partial s^2}\ ds \qquad (5)$$

And then using variation of parameter method to solve the resulting non homogeneous system of ODEs which is known as method of lines.

2. Method of Lines

2.1 Discretization

To approximate the solution of eq- (14) we define mesh points x_j as $x_j = a + j\Delta x$, $j = 0,1, \dots, N$,where Δx shows spatial step size also $N = (b-a)/\Delta x$.

By substitution $x = x_j$ ($= 1, 2, \dots, N-1$) into Eq. (1) we have

$$\frac{\partial u\,(x_j,t)}{\partial t} = d(x_j,t)\,\frac{\partial^\alpha u\,(x_j,t)}{\partial x^\alpha} + p\,\left(x_j,t,u(x_j,t)\right) \qquad (6)$$

Now $\frac{du_j}{dt} = d(x_j,\ t)\,\frac{\partial^\alpha u(x_j,\,t)}{\partial x^\alpha} + p(x_j,\ t, u(x_j,\ t))$, $(x_j,\ t) \in \Omega$

For $\frac{\partial^\alpha u(x_j,\,t)}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)}\ \int_a^{x_j}\ (x_j - s)^{1-\alpha}\ .\ \frac{\partial^2 u(s,t)}{\partial s^2}\ ds$

$= \frac{1}{\Gamma(2-\alpha)}\ \int_0^{x_j-a}\ (s)^{1-\alpha}\ .\ \frac{\partial^2 u\,(x_j-s\,,\ t)}{\partial s^2}\ ds$ let $s = x_j - s$

$= \frac{1}{\Gamma(2-\alpha)}\ \sum_{k=0}^{j-1}\ \int_{x_k}^{x_{k+1}} s^{1-\alpha}\ .\ \frac{\partial^2 u\,(x_j-s\,,\ t)}{\partial s^2}\ ds\,, \quad j = 1,2,\dots,N-1$

$= \frac{1}{\Gamma(2-\alpha)}\ \sum_{k=0}^{j-1}\ \int_{kh}^{(k+1)h} s^{1-\alpha}\ .\ \frac{\partial^2 u\,(x_j-s\,,\ t)}{\partial s^2}\ ds$

$= \frac{1}{\Gamma(2-\alpha)}\ \sum_{k=0}^{j-1} \frac{u(x-(k-1)h)-2u(x-kh)+u(x-(k+1)h)}{h^2}\ \int_{kh}^{(k+1)h} s^{1-\alpha}\,ds \qquad (7)$

Now for the integration part in eq (7)

$$\int_{kh}^{(k+1)h} s^{1-\alpha}\,ds = \left[\frac{s^{2-\alpha}}{2-\alpha}\right]_{kh}^{(k+1)h}$$

$$= \frac{1}{2-\alpha}\ \left[\left((k+1)h\right)^{2-\alpha} - (kh)^{2-\alpha}\right]$$

$$\int_{kh}^{(k+1)h} s^{1-\alpha}\,ds = \frac{h^{2-\alpha}}{2-\alpha}\ \left((k+1)^{2-\alpha} - k^{2-\alpha}\right) \qquad (8)$$

Substituting value of result (8) in eq (7) we get,

$$= \frac{h^{-\alpha}}{\Gamma 3-\alpha}\ \sum_{k=0}^{j-1}[u(x-(k-1)h) - 2u(x-kh) + u(x-(k+1)h][(k+1)^{2-\alpha} - k^{2-\alpha}] \qquad (9)$$

Substituting value of result (9) in eq (6) we get,

$$= d(x_j,t)\frac{h^{-\alpha}}{\Gamma 3-\alpha}\ *$$

$$\sum_{k=0}^{j-1}\ [\,u(x-(k-1)h) - 2u(x-kh) + u(x-(k+1)h\,]\ [(k+1)^{2-\alpha} - k^{2-\alpha}]$$

$$+p\,\left(x_j,t,u(x_j,t)\right) \qquad (10)$$

where $j = 1,2,\dots,N-1$

Here we got linear system of ODE.

Now using the variation of parameter method to solve the above non homogeneous system of ODEs and method of lines we find the solution of equation.

2.2 Variation of parameter method

The **variation of parameters** is a general method used to find a particular solution for non-homogeneous linear ordinary differential equations (ODEs). It is especially useful when the non-homogeneous term is complex.

Consider the linear system (10) in the matrix form

$$u'(t) = A(t)u(t) + F(t) \quad (11)$$

Where,

$u(t)$ is an $(N-1) \times 1$ vector,

$A(t)$ is an $(N-1) \times (N-1)$ matrix,

F(t) is the forcing (non-homogeneous) vector.

The variation of parameters method gives the complete solution.

Suppose the fundamental matrix of the matrix $A(t)$ solution is

$$\Phi(t) = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_{N-1} \\ | & & | \end{bmatrix}$$

where the columns are linearly independent solutions.

Then the homogeneous solution is $u_h(t) = \Phi(t)C.$

where C is a constant vector.

Now, Assume $u_p(t) = \Phi(t)y(t).$

where $y(t)$ is an unknown vector function. After solving we get $u_p(t) = \Phi(t)\int \Phi^{-1}(t)F(t)dt.$ Which is the variation of parameters formula.

2.3 Method of lines

The method of lines [13,14,15] is a technique for solving partial differential equations (PDEs) in which all but one dimension is discretized. By reducing a PDE to a single continuous dimension, the method of lines allows solutions to be computed via methods and software developed for the numerical integration of ordinary differential equations (ODEs) and differential-algebraic systems of equations (DAEs). Many integration routines have been developed over the years in many different programming languages, and some have been published as open source resources[16].

The method of lines most often refers to the construction or analysis of numerical methods for partial differential equations that proceeds by first discretizing the spatial derivatives only and leaving the time variable continuous. This leads to a system of ordinary differential equations to which a numerical method for initial value ordinary equations can be applied. The method of lines in this context dates back to at least the early 1960s [17]. Many papers discussing the accuracy and stability of the method of lines for various types of partial differential equations have appeared since [18,19].

3. Numerical examples

To show computational efficiency of the present method (MOL), some selected test problems including SFDEs with linear source terms are solved numerically. All of the computations are performed with MATLAB.

Example-1

The following SFDE has the exact solution which can be used to verify the numerical solution [9]

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{1.5}u(x,t)}{\partial x^{1.5}} + p(x,t) \quad , 0 < x < 1$$

With the diffusion coefficient $d(x) = \Gamma(1.5)x^{0.5}$, and the source term

The initial condition is $u(x,0) = (x^2+1)\sin(1)$,

And the Dirichlet boundary conditions are

$u(0,t) = \sin(t+1)$ and $u(1,t) = 2\sin(t+1)$, $t > 0$

The exact solution of this test example is $u(x,t) = (x^2+1)\sin(t+1)$.

Table-1 Comparison of absolute errors at t=1 for Example 3.1 (Δx= Δt=0.1)

xx	SLT [9]	Sousa [11]	Salehi [1]	Our method
0.1	4.66e -05	2.58e -05	2.96e -07	4.219e -15
0.2	7.74e -05	3.23e -05	4.47e -07	6.772e -15
0.3	5.00e -05	4.02e -05	5.67e -07	6.883e -15
0.4	2.30e -05	4.54e -05	6.02e -07	6.883e -15
0.5	2.74e -05	4.87e -05	5.74e -07	4.885e -15
0.6	4.38e -05	4.89e -05	4.88e -07	4.663e -15
0.7	3.87e -05	4.52e -05	3.68e -07	1.110e -15
0.8	1.01e -05	3.66e -05	2.69e -07	1.110e -15
0.9	3.35e -05	2.19e -05	1.02e -07	0.444e -15

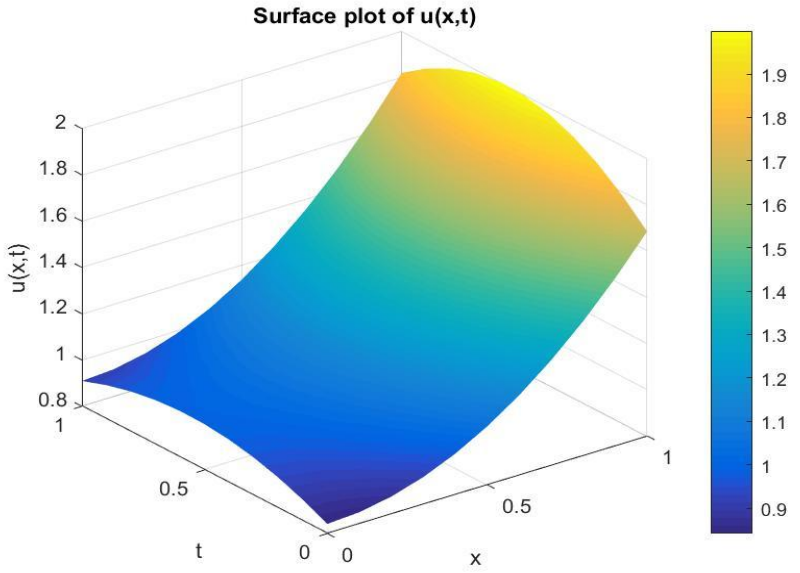


Fig-1 Solution u(x, t) graph of Example-1 for Δ x = Δ t = 0.1

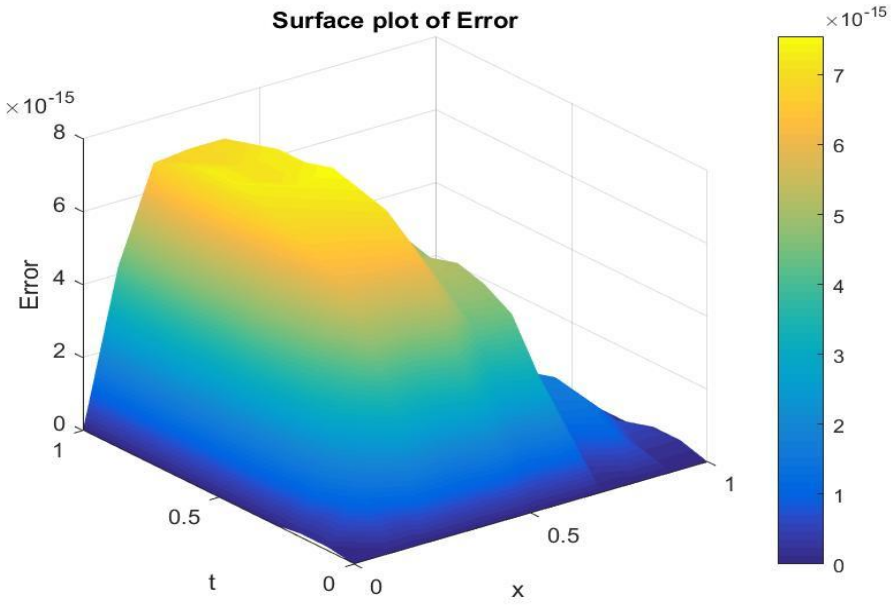


Fig-2 Graph for Comparison of absolute errors for Δ x = Δ t = 0.1

We compare MOL solution with some past works in the Table 1 with $\Delta x = 0.1$ and $\Delta t = 0.1$ for the Example 3.1, which gives that proposed method is the most accurate one. The MOL approximate solution and the absolute errors are displayed in Figures 1 and 2, respectively.

Example-2

Now the SFDE for second test example is ,

$$\frac{\partial u\,(x\,,t)}{\partial t} = \frac{24\,x^{\alpha}}{\Gamma\,(5+\alpha)}\;\frac{\partial^{\alpha} u\,(x,t)}{\partial x^{\alpha}} - 2\,u\,(x,t)\;\;,\; 0 < x < 1$$

The initial condition $u\,(x,0) = x^{4+\alpha}$ and

Boundary conditions are $u\,(\,0,t) = 0\;\;,\;\; u\,(\,1\,,\,t) = e^{-t}\;\;,\; t > 0$

The exact solution of this example is $u\,(\,x\,,t\,) = e^{-t}\;x^{4+\alpha}$

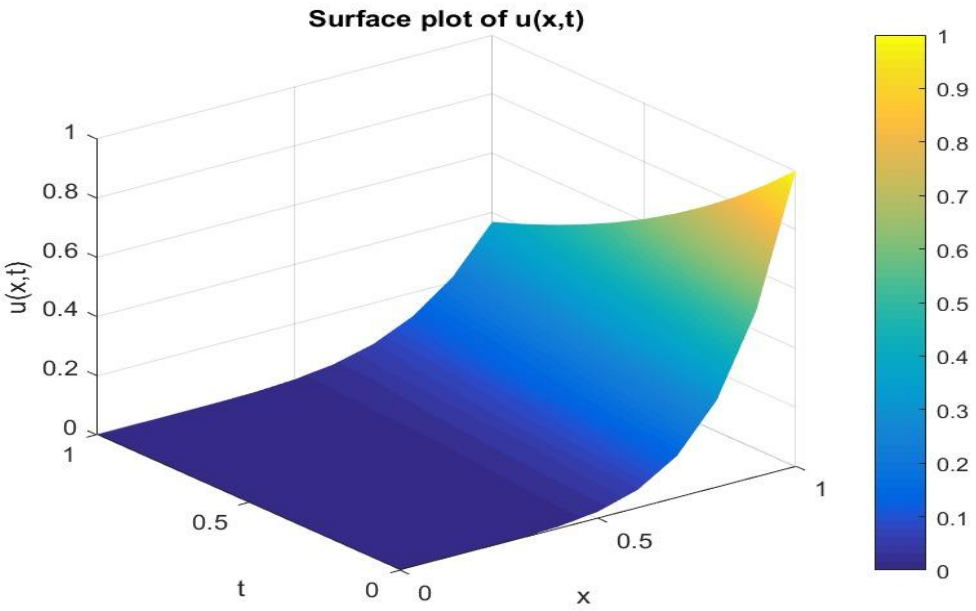


Fig-3 Solution graph u(x,t) for example-2 for Δx = Δt = 0.1

The numerical results obtained by the proposed Method of Lines (MOL) scheme show very good agreement with the exact solution. The surface plot of the approximate solution is presented in Figure 3, which demonstrates the stability and accuracy of the proposed numerical method.

3.2 Numerical example-3

The SFDE is,

$\frac{\partial u(x,t)}{\partial t} = d(x)\ \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + p(x,t)\ ,\quad 0 < x < 2$

where $d(x) = \Gamma(4-\alpha)\,.\,x^{\alpha}$ and $p(x,t) = e^{-t}x^{2}\,(7x + 2\alpha - 7)$

Initial condition $u(x,0) = x^{2} - x^{3}$

Neumann boundary conditions are $u_x\,(0\,,\,t) = 0\quad,\quad u_x\,(2\,,\,t) = -8\,e^{-t}$

The exact solution of this problem is $u\,(x\,,\,t) = (x^{2} - x^{3})\,e^{-t}$

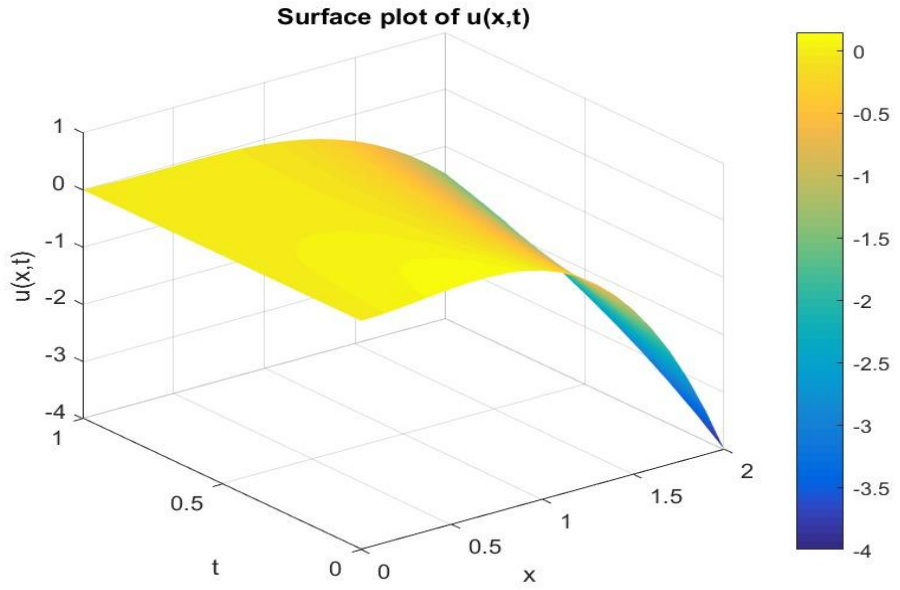


Fig-4 Solution graph u(x,t) for example-3 for $\Delta x = \Delta t = 0.1$

For the third test example with Neumann boundary conditions, the proposed Method of Lines (MOL) approach is applied to obtain the numerical solution of the SFDE. The computed results are found to be in close agreement with the corresponding exact solution, confirming the effectiveness and reliability of the proposed numerical technique. The approximate behavior of the solution over the spatial and temporal domains is illustrated through the surface plot in Figure 4. The graphical representation demonstrates that the MOL scheme successfully handles the Neumann boundary conditions while maintaining good numerical stability and accuracy throughout the computational region.

4. Conclusion

In this work, an efficient Method of Lines (MOL) technique has been successfully applied for solving space-fractional differential equations under different types of boundary conditions. The fractional spatial derivative was discretized by an appropriate finite difference approximation, which transformed the original SFDE into a system of ordinary differential equations. The resulting system was then solved using the variation of parameter method, thereby converting the proposed approach into a semi-analytical Method of Lines technique. The proposed method was tested on several examples involving both Dirichlet and Neumann boundary conditions. The obtained numerical solutions showed excellent agreement with the corresponding exact solutions, confirming the validity and effectiveness of the present approach. The surface plots further illustrate the smooth behavior, stability, and accuracy of the numerical scheme over the computational domain.The numerical experiments demonstrate that the semi-analytical MOL approach is simple to implement, computationally efficient, and capable of handling fractional-order problems with high accuracy.

REFERENCES

1. Younes Salehi, Mo. T. Darvishi , Wiliam E. Schiesser , Numerical solution of space fractional diffusion eq by method of lines and splines
2. W. Deng, Finite element method for the space and time fractional Fokker–Planck equation, SIAM J. Numer. Anal. 47 (1) (2008) 204–226.
3. V.J. Ervin, N. Heuer, J.P. Roop, Numerical approximation of a time dependent, nonlinear, space-fractional diffusion equation, SIAM J. Numer. Anal. 45 (2) (2007) 572–591.
4. F. Liu, V. Anh, I. Turner, Numerical solution of the space fractional Fokker–Planck equation, J. Comput. Appl. Math. 166 (1) (2004) 209–219.

5. S.S. Ray, Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method, Commun. Nonlinear Sci. Numer. Simul. 14 (4) (2009) 1295–1306.
6. J. Sabatier , O.P. Agrawal , J.T. Machado , Advances in Fractional Calculus, Springer, 2007.
7. [7] M. Khader, Onthe numerical solutions for the fractional diffusion equation, Commun. Nonl. Sci. Numer. Simul. 16 (6) (2011) 2535–2542.
8. [8] B. Zhu, The space fractional diffusion equation with Feller's operator, Appl. Math. Comput. 217 (24) (2011) 9971–9977.
9. [9] A. Saadatmandi, M. Dehghan, A Tau approach for solution of the space fractional diffusion equation, Comput. Math. Appl. 62 (3) (2011) 1135–1142.
10. [10] R.F. Ren, H.B. Li, W. Jiang, M.Y. Song, An efficient Chebyshev-Tau method for solving the space fractional diffusion equations, Appl. Math. Comput. 224 (2013) 259–267.
11. [11] E. Sousa, Numerical approximations for fractional diffusion equations via splines, Comput. Math. Appl. 62 (3) (2011) 938–944.
12. Diethelm, Kai (2019). "General theory of Caputo-type fractional differential equations". Fractional Differential Equations. pp. 1–20. doi:10.1515/9783110571660-001. ISBN 978-3-11-057166-0. Retrieved 2023-08-10
13. Schiesser, W. E. (1991). The Numerical Method of Lines. Academic Press. ISBN 0- 12-624130-9.
14. Hamdi, S.; W. E. Schiesser; G. W. Griffiths (2007), "Method of lines", Scholarpedia, 2 (7):2859, Bibcode:2007SchpJ...2.2859H, doi:10.4249/scholarpedia.2859
15. Schiesser, W. E.; G. W. Griffiths (2009). A Compendium of Partial Differential Equation Models: Method of Lines Analysis with MATLAB. Cambridge University Press. ISBN 978-0-521-51986-1.
16. Lee, H. J.; W. E. Schiesser (2004). Ordinary and Partial Differential Equation Routines in C, C++, Fortran, Java, Maple and MATLAB. CRC Press. ISBN 1-58488-423-1.
17. E. N. Sarmin; L. A. Chudov (1963), "On the stability of the numerical integration of systems of ordinary differential equations arising in the use of the straight line method", USSR Computational Mathematics and Mathematical Physics, **3** (6): 1537–1543, doi:10.1016/0041-5553(63)90256-8
18. A. Zafarullah (1970), "Application of the Method of Lines to Parabolic Partial Differential Equations With Error Estimates", Journal of the Association for Computing Machinery, vol. 17, no. 2, pp. 294–302, doi:10.1145/321574.321583, S2CID 15114435
19. J. G. Verwer; J. M. Sanz-Serna (1984), "Convergence of method of lines approximations to partial differential equations", Computing, **33** (3–4): 297–313, doi:10.1007/bf02242274, S2CID 30171258
20. P. Saucez, W. Schiesser and All Authors, Adaptive method of lines, CRC Press, 2001.
21. T. Koto, Method of lines approximations of delay differential equations, Computers & Mathematics with Applications, 48(1-2) (2004), 45-59.
22. S. Hamdi, W. Schiesser and G. Grifths, Method of lines, Scholarpedia, 2(7) (2007), 2859.
23. G. Patel and K. Patel, The method of lines for solution of one dimensional heat equation, Proceeding of the International Conference on Emerging Trends in Scientific Research, 1 (2015), 200-208.
24. G. Patel and K. Patel, The method of lines for solution of the one dimensional second order wave equation,Mathematical Sciences International Research Journal, 4(2) (2015), 98-104.
25. G. Patel and K. Patel, The method of lines for solution of the two dimensional elliptic equation, Annals of FacultyEngineeeringHunedoaraInternational Journal of Enginnering, XIV(1) (2016), 225-230.